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Resolutions of nilpotent orbit closures via Coulomb branches of 3-dimensional $\mathcal{N} = 4$ theories

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Abstract: The Coulomb branches of certain 3-dimensional $\mathcal{N} = 4$ quiver gauge theories are closures of nilpotent orbits of classical or exceptional Lie algebras. The monopole formula, as Hilbert series of the associated Coulomb branch chiral ring, has been successful in describing the singular hyper-Kähler structure. By means of the monopole formula with background charges for flavour symmetries, which realises real mass deformations, we study the resolution properties of all (characteristic) height two nilpotent orbits. As a result, the monopole formula correctly reproduces (i) the existence of a symplectic resolution, (ii) the form of the symplectic resolution, and (iii) the Mukai flops in the case of multiple resolutions. Moreover, the (characteristic) height two nilpotent orbit closures are resolved by cotangent bundles of Hermitian symmetric spaces and the unitary Coulomb branch quiver realisations exhaust all the possibilities.

Keywords: Global Symmetries, Supersymmetric Gauge Theory, Differential and Algebraic Geometry, Field Theories in Lower Dimensions

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1 Introduction

Nilpotent orbits play an important role for supersymmetric gauge theories and appear whenever an embedding of SU(2) into some group is involved. In particular, nilpotent orbit closures have become the prototypical example of non-trivial hyper-Kähler singularities which can be realised as Coulomb and Higgs branches of 3-dimensional $\mathcal{N} = 4$ gauge theories. This prominent role of nilpotent orbit closure is due to a theorem by Namikawa [1].

The realisation that Coulomb $\mathcal{M}_C$ and Higgs $\mathcal{M}_H$ branches can capture diverse features of the geometry of nilpotent orbits and Slodowy slices was due to the consideration of boundary conditions in $\mathcal{N} = 4$ supersymmetric gauge theories [2]. The Higgs and Coulomb branch realisations of nilpotent orbits have been systematically developed in [3, 4]. Recently [5, 6], another geometrical phenomenon has been realised in the Type IIB superstring set-up of 3-dimensional $\mathcal{N} = 4$ theories: the classification of transverse slices and their singularities can be engineered by the *Kraft-Procesi* transition, which is nothing else than a

Higgs mechanism. Here, the partial ordering of nilpotent orbit closures and the notion of transverse slice allow to study the singularity structures of $\mathcal{M}_C$ and $\mathcal{M}_H$. Uplifting this procedure to generic 3-dimensional $\mathcal{N} = 4$ quiver gauge theories has turned out to be fruitful and is called *quiver subtraction* [7].

We take the recent advances as motivation to investigate the resolutions of the Coulomb branches which correspond to height two nilpotent orbit closures of classical and exceptional Lie algebras. The Coulomb branch geometry is affected by two types of deformations: complex and real mass deformations. Complex mass deformations are known to lead to deformations of $\mathcal{M}_C$ and, interestingly, exhibit another part of the geometry of nilpotent orbits: they are geometrical incarnations of *induced nilpotent orbits* [8]. Moreover, complex mass deformations are accommodated for in the *abelianisation* approach to $\mathcal{M}_C$ of [9]. However, this approach is insensitive to real mass deformations. In contrast, the Hilbert series of the Coulomb branch [10] is sensitive to real mass deformations, but not to complex masses. It is expected that real mass deformations (at least partially) resolve the Coulomb branch singularities.

As pioneered by [10–12], $\mathcal{M}_C$ and $\mathcal{M}_H$ can be studied as algebraic varieties via the Hilbert series, which counts gauge invariant chiral operators on the associated chiral ring.[1] The corresponding Hilbert series is called *monopole formula* [10] because the relevant operators to consider on the chiral ring are monopole operators. The prescription of the monopole formula is capable to accommodate discrete real mass parameters via background charges (fluxes) for the flavour symmetry [13]; hence, one can utilise the Coulomb branch Hilbert series to study resolutions of the conical singularities.

The idea of introducing background charges into the monopole formula is not new, as it found applications in gluing techniques [13, 14] or have been shown to be exchanged with baryonic background charges in the Hilbert series [15, 16] of $\mathcal{M}_H$ upon mirror symmetry in [17]. However, to the best of our knowledge, it has not been used to study resolutions of Coulomb branches systematically. Explicit examples include 3-dimensional $\mathcal{N} = 4$ SQED in [18, section 2.5.2] and the study [19] of cotangent bundles $T^*(G/H)$ of Kähler cosets, which are known to appear in the Springer resolution of nilpotent orbit closures. The latter approached the Hilbert series of $T^*(G/H)$ neither from the monopole formula nor with background charges present. Moreover, resolutions of classical nilpotent orbit closures, realised by the so called $T^\sigma_\rho(G)$ theories, have been discussed in [13] and the information has been employed to derive the general Hilbert series in terms of Hall-Littlewood polynomials.

The outline of the remainder is as follows: in section 2 we recall the relevant concepts such as nilpotent orbits and their symplectic resolutions as well as 3-dimensional $\mathcal{N} = 4$ gauge theories, and the monopole formula. Thereafter, we systematically consider all (characteristic) height two orbits of classical algebras in sections 3–6 and for exceptional algebras in section 7. We conclude and summarise in section 8. The conventions for the calculations are provided in appendix A.

[1] We would like to emphasise that there is no need to (being able to) pick an entire sub algebra of the supersymmetry algebra; it is enough to pick a complex linear combination of supercharges to define the notion of chirality. This is crucial for 5 and 6 dimensional theories with 8 supercharges.

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2 Preliminaries

2.1 Nilpotent orbits

As we are concerned with quiver gauge theories whose Coulomb branches are closures of nilpotent orbits, we review the necessary ingredients. A general reference for nilpotent orbits is [20].

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and G its adjoint group. We recall that nilpotent orbits $\mathcal{O} \subset \mathfrak{g}$, as complex adjoint orbits of G, are equipped with a canonical holomorphic symplectic from, the *Kirillov-Kostant-Souriau form.* The fact that the adjoint orbits are hyper-Kähler varieties has been proven by Kronheimer [21, 22] in important special cases and by Biquard [23] and Kovalev [24] in full generality.

The set of nilpotent orbits of $\mathfrak{g}$ is finite and the set of orbit closures admits a partial ordering via inclusion. Besides the trivial orbit, there exists a unique minimal orbit $\mathcal{O}_{\min}$, whose closure is contained in the closure of any other non-trivial nilpotent orbit. In addition, there exists a unique maximal orbit $\mathcal{O}_{\max}$, whose closure contains any other orbit closure. The closure $\overline{\mathcal{O}}_{\max}$ is called the nilpotent cone. Due to the nilpotency condition, any nilpotent orbit $\mathcal{O}$ is invariant under the dilation action of $\mathbb{C}^\times$ on $\mathfrak{g}$.

Classical algebras. Nilpotent orbits for the classical Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{sp}(n)$, and $\mathfrak{so}(n)$, can be labelled by partitions $\rho = (d_1, \ldots, d_t)$ of some $N \in \mathbb{N}$, with $d_1 \geq d_2 \geq \ldots \geq d_t$ and $\sum_{i=1}^t d_i = N$. Let $\mathcal{N}(\mathfrak{g})$ be the finite set of nilpotent orbits of $\mathfrak{g}$, then the following holds [25, Proposition 2.1]:

(A_n) If $\mathfrak{g} = \mathfrak{su}(n+1)$, then there exists a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions ρ of $n+1$.

(B_n) If $\mathfrak{g} = \mathfrak{so}(2n+1)$, then there exists a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions ρ of $2n+1$ such that even parts have even multiplicity.

(C_n) If $\mathfrak{g} = \mathfrak{sp}(2n)$, then there exists a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions ρ of $2n$ such that odd parts have even multiplicity.

(D_n) If $\mathfrak{g} = \mathfrak{so}(2n)$, then there exists a surjection f from $\mathcal{N}(\mathfrak{g})$ to the set of partitions ρ of $2n$ such that even parts have even multiplicity. A partition of even parts only is called very even. For ρ not a very even partition, $f^{-1}(\rho)$ consists of exactly one orbit. For ρ very even, $f^{-1}(\rho)$ consists of exactly two different orbits.

Alternatively, nilpotent orbits may be labelled by *weighted Dynkin diagrams*, which are briefly discussed in appendix B. Next, we recall the *height* $\mathrm{ht}(\mathcal{O}_\rho)$ of $\mathcal{O}_\rho$ from [26]:

(i) $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$ then $\mathrm{ht}(\mathcal{O}_\rho) = 2(d_1 - 1)$

(ii) $\mathfrak{g} = \mathfrak{so}(n)$ then $$\mathrm{ht}(\mathcal{O}_\rho) = \begin{cases} d_1 + d_2 - 2, & d_2 \geq d_1 - 1 \\ 2d_1 - 4, & d_2 \leq d_1 - 2 \end{cases}$$

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In this work, we restrict to nilpotent orbits of $\mathrm{ht}(\mathcal{O}) = 2$ for the following two reasons: (i) the closure of a nilpotent orbit of height $\mathrm{ht}(\mathcal{O}) \leq 2$ always admits a unitary Coulomb branch quiver realisation[2] and (ii) the observation from [3, 4] that height two orbit closures have a simple Hilbert series or Highest Weight Generating function. The details will become clear in the subsequent sections.

Exceptional algebras. The classification of nilpotent orbits for exceptional algebras is more involved. One obvious obstacle to overcome is that exceptional groups do not act as matrices on their fundamental vector space. Several labelling methods for nilpotent orbits have been developed by Dynkin [27], Bala-Carter [28, 29], and Hesselink [30], to name a few. Here, with the aim to study Coulomb branches, we follow the labelling by Characteristics and refer for details to [4]. The height of exceptional nilpotent orbits can be calculated following [26, section 2]; the definition agrees for classical algebras with the partition data given above.

2.2 Resolutions

It is well-known that the closure of nilpotent orbit $\mathcal{O}$ is a singular (in general non-normal) variety. By Hironaka's work [31, 32], any complex variety admits a resolution, but there may exist many different resolutions. One would like to restrict to certain "good" resolutions. For a symplectic variety, such a preferred resolution has been introduced by Beauville [33], denoted as *symplectic resolution*. Roughly, for a symplectic variety X and $\pi : Z \to X$ a resolution, then π is a symplectic resolution if for any symplectic form ω on the regular part of X, the pull-back $\pi^*(\omega)$ extends to a symplectic form on Z.

For nilpotent orbits, it was proven by Panyushev [34] that the natural symplectic 2-form on $\mathcal{O}$ extends to any resolution of $\overline{\mathcal{O}}$, i.e. $\overline{\mathcal{O}}$ is a variety with a symplectic singularity. Subsequently, Fu [35] determined all nilpotent orbit closures which admit a symplectic resolution.

One finds that every nilpotent orbit closure in $\mathfrak{sl}(n+1)$ admits a symplectic resolution [36, Proposition 5.1]. However, if one hopes that every nilpotent orbit closure admits a resolution, one finds the following disillusioning result [36, Proposition 5.2]: for a simple Lie algebra $\mathfrak{g}$, the closure $\overline{\mathcal{O}}_{\mathrm{min}}$ admits a symplectic resolution if and only if $\mathfrak{g}$ is of A-type. The general statement [35] is that a symplectic variety $\overline{\mathcal{O}}$ admits a symplectic resolution if and only if $\mathcal{O}$ is a *Richardson orbit*, see [20] for a definition. For the orbits which do admit a symplectic resolution, it is of the form [35]

$$\pi : Z \to \overline{\mathcal{O}} \qquad \text{with} \quad Z \cong T^* \left(G/P\right) \tag{2.1}$$

where $P \subset G$ is some parabolic subgroup — called *polarisation* of $\mathcal{O}$, see [30].

Even when restricting the attention to symplectic resolutions $T^*(G/P) \to \overline{\mathcal{O}}$, there can exist several polarisations which yield different symplectic resolutions $T^*(G/P_i) \to \overline{\mathcal{O}}$. The rational map $\phi : T^*(G/P_1) \dashrightarrow T^*(G/P_2)$ between any two resolutions is called a locally trivial family of *Mukai flops*. Following [25, 37], there are three basic types of Mukai flops: A, D, and $E_{6,I}/\ E_{6,II}$. For a generic $\overline{\mathcal{O}}$, the birational map ϕ decomposes

[2] See a comment in [7, Footnote 11].

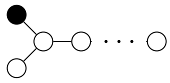
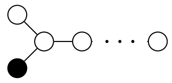
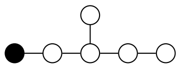
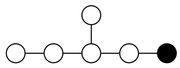
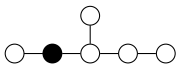
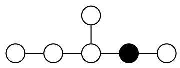
 k	A_n with $2k+1 \neq n$	 $n{+}1{-}k$
	D_n with $n = \text{odd}$	
	$E_{6,I}$	
	$E_{6,II}$	

Table 1. List of dual marked Dynkin diagrams for the dual parabolic subgroups leading to the Mukai flops of type A, D, $E_{6,I}$, and $E_{6,I}$.

into a finite number of diagrams $Y_l \to X_l \leftarrow Y_{l+1}$, $l = 1, \ldots, m$, with $Y_1 = T^*(G/P_1)$ and $Y_m = T^*(G/P_2)$ such that each diagram is a basic Mukai flop. For closures of height two nilpotent orbits the basic Mukai flops suffice.

Mukai flop of type *A*. Let $x \in \mathfrak{su}(n{+}1)$ be a nilpotent element of partition $(2^k, 1^{n+1-2k})$ and $x \in \mathcal{O}$. Then there exist two polarisations P and P' of x such that $P = \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n+1-k))$ and $P' = \mathrm{S}(\mathrm{U}(n+1-k) \times \mathrm{U}(k))$. Hence, $\overline{\mathcal{O}}$ admits two Springer resolutions

$$T^*\left(\mathrm{SU}(n+1)/P\right) \xrightarrow{\pi} \overline{\mathcal{O}} \xleftarrow{\pi'} T^*\left(\mathrm{SU}(n+1)/P'\right) . \tag{2.2}$$

Note that the dual parabolic subgroups P, P' can be read off from the marked Dynkin diagrams in table 1. If $2k < n+1$ then (2.2) is a flop [25, Lemma 3.1]; and if $2k = n+1$ then the two resolutions are isomorphic [25, Remark 3.2].

Mukai flops of type *D*. Suppose $n = \text{odd}$ and $n \geq 3$. Let $x \in \mathfrak{so}(2n)$ be a nilpotent element of type $(2^{n-1}, 1^2)$ and $x \in \mathcal{O}$. Then there exist two choices of flags P_+ and P_-; hence, $\overline{\mathcal{O}}$ admits two Springer resolutions

$$T^*(\mathrm{SO}(2n)/P_+) \xrightarrow{\pi_+} \overline{\mathcal{O}} \xleftarrow{\pi_-} T^*(\mathrm{SO}(2n)/P_-) \,. \tag{2.3}$$

From the marked Dynkin diagrams in table 1, we read off the dual parabolic subgroups to be $P_\pm \cong \mathrm{SU}(n) \times \mathrm{U}(1)$. The $\pm$ subscript refers to the choice of the spinor node.

Mukai flops of type $\boldsymbol{E_6}$. The $E_{6,I}$ corresponds to the orbit with Bala-Carter label $2A_1$ or the Characteristic $\{1,0,0,0,1,0\}$. The two dual parabolic subgroup $\cong \mathrm{SO}(10) \times \mathrm{SO}(2)$ can be read off from the marked Dynkin diagrams in table 1.

The basic flop $E_{6,II}$ corresponds to the orbit with Bala-Carter label $A_2 + 2A_1$ or Characteristic $\{0,1,0,1,0,0\}$. From the marked Dynkin diagrams in table 1, we can read off the two dual parabolic subgroups $\cong \mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

Hermitian symmetric spaces. Since (characteristic) height two nilpotent orbit closures are discussed below, it is useful to recall the Hermitian symmetric spaces (HSS). The HSS were first classified by Cartan [38] and can be realised as homogeneous spaces G/H, see table 2. In terms of Cartan's classification of compact Riemannian symmetric spaces,

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name	G	H	$\dim_{\mathbb{C}}(G/H)$
A_{III}	$\mathrm{SU}(n+m)$	$S(\mathrm{U}(n)\times\mathrm{U}(m))$	$n\cdot m$
D_{III}	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$\frac{1}{2}n(n-1)$
C_I	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$\frac{1}{2}n(n+1)$
B_I/D_I	$\mathrm{SO}(n+2)$	$\mathrm{SO}(n)\times\mathrm{U}(1)$	n
E_{III}	E_6	$\mathrm{SO}(10)\times\mathrm{U}(1)$	16
E_{VII}	E_7	$E_6\times\mathrm{U}(1)$	27

Table 2. The four infinite series and the two exceptional cases of the Hermitian symmetric spaces.

the Hermitian symmetric spaces are the four infinite series A_{III}, D_{III}, C_I, B_I/D_I, and the two exceptional spaces E_{III}, E_{VII}.

As a remark, since a HSS of the form G/H is Hermitian as well as a symmetric homogeneous space, it follows that G/H is also Kähler. Consequently, $T^*(G/H)$ is naturally hyper-Kähler and we will encounter the cotangent bundles of the HSS spaces below.

2.3 3-dimensional $\mathcal{N}=4$ gauge theories

We consider a generic 3-dimensional $\mathcal{N}=4$ gauge theory with gauge group $\mathcal{G}$ and matter, in the form of hypermultiplets, transforming in some (quaternionic) representation $\oplus_I n_I \mathcal{R}_I$ of $\mathcal{G}$. Depending on the multiplicities n_I there exists a non-trivial flavour symmetry G_F, sometimes called "Higgs branch" global symmetry. In addition, if $\mathcal{G}$ contains abelian factors, there exists another global "Coulomb branch" symmetry G_J which in the ultraviolet is given by $G_J^{\mathrm{UV}} = \mathrm{U}(1)^{\#(\mathrm{U}(1) \text{ in } \mathcal{G})}$, and may be enhanced in the infrared to a non-abelian group G_J^{IR} whose maximal torus is at least[3] G_J^{UV}. In addition, there is a non-trivial R-symmetry group $\mathrm{SU}(2)_{\mathcal{C}} \times \mathrm{SU}(2)_{\mathcal{H}}$, such that the three vector multiplet scalars are a triplet under $\mathrm{SU}(2)_{\mathcal{C}}$ and the hypermultiplets transform as doublets under $\mathrm{SU}(2)_{\mathcal{H}}$.

In the absence of mass deformations, the vacuum moduli space has a rich structure as a union of several branches of the form $\cup_a \mathcal{C}_a \times \mathcal{H}_a$. $\mathcal{C}_a$ is a hyper-Kähler space parametrised by vacuum expectation values (VEVs) of gauge-invariant combinations of vector multiplet scalars; whereas $\mathcal{H}_a$ is a hyper-Kähler space parametrised by VEVs of gauge invariant combinations of the hypermultiplet scalars. The *Coulomb branch* $\mathcal{M}_C$ and *Higgs branch* $\mathcal{M}_H$ arise as maximal branches with one factor being trivial. The global symmetries G_F and G_J act on $\mathcal{M}_H$ and $\mathcal{M}_C$, respectively. Moreover, these actions are associated to triplets of moment maps. Geometrically, $\mathcal{M}_C$ and $\mathcal{M}_H$ are hyper-Kähler singularities with $\mathrm{SU}(2)_{\mathcal{C}}$ or $\mathrm{SU}(2)_{\mathcal{H}}$ isometry, respectively.

The 3-dimensional gauge theories with eight supercharges allow for two classes of deformation parameters: masses and FI parameters, which take values in a Cartan subalgebra of G_F and G_J, respectively. Under the R-symmetry, the masses transform as triplet under $\mathrm{SU}(2)_{\mathcal{C}}$ and the FI parameters form a triplet under $\mathrm{SU}(2)_{\mathcal{H}}$. It is a known feature that

[3]In the class of examples considered here, the ranks of UV and IR Coulomb branch global symmetry coincide.

masses can deform and/or resolve the geometry of the Coulomb parts $\mathcal{C}_a$; while FI parameters deform / resolve the Higgs parts $\mathcal{H}_a$. Restricting attention to the Coulomb branch, the triplet $\{m_i\}_{i=1}^3$ of masses decomposes into a *complex mass* $m^{\mathbb{C}} = m_1 + \mathrm{i}m_2$ and a *real mass* $m^{\mathbb{R}} = m_3$. The two have different implications on the geometry: while the real mass leads to a (partial) resolution of the singularities of $\mathcal{M}_C$, the complex mass will deform the geometry of $\mathcal{M}_C$.

Starting from [2], it has been realised that nilpotent orbit closures appear as Coulomb and Higgs branches of 3-dimensional $\mathcal{N} = 4$ gauge theories. The relation between quiver graphs and nilpotent orbit closures has been established in the mathematics literature by at least [39, 40]. These mathematical construction are all on the Higgs branch. Recently, Namikawa [1] proved the following: if all generators of a hyper-Kähler singularity with an $\mathrm{SU}(2)_R$ symmetry have spin $= 1$ under $\mathrm{SU}(2)_R$, then the corresponding variety is a nilpotent orbit closure of the Lie algebra of its isometry group. Consequently, nilpotent orbit closures are to be considered as the simplest non-trivial singular hyper-Kähler spaces.

In the following, we will consider quiver gauge theories with unitary gauge groups, which have enhanced non-abelian global symmetry G_J^{IR} of $ABCDE$-type, in order to realise nilpotent orbit closures of $\mathrm{Lie}(G_J^{\mathrm{IR}})$. For quiver theories, one reads off the Dynkin diagram of the non-abelian part of G_J^{IR} from the set of *balanced nodes*. Recall, a gauge node is balanced if the number of flavours is equal to twice its rank.

2.4 Coulomb branch realisations of nilpotent orbit closures

In [2] a class of 3-dimensional superconformal field theories, denoted as $T_\sigma^\rho(G)$, has been introduced. These theories arise as infrared limits of linear quiver gauge theories with unitary or alternating orthogonal-symplectic gauge groups. Here, G is considered as classical group with GNO dual $\widehat{G}$ [41]; ρ is a partition of G and σ is a partition of $\widehat{G}$, as defined above. By construction, the mirror of $T_\sigma^\rho(G)$ is $T_\rho^\sigma(\widehat{G})$. For classical G, all these theories can be seen as originating from Type IIB brane constructions.

In this work, we restrict ourself to the Coulomb branch of $T^\rho(G)$ theories, which are obtained from $T_\sigma^\rho(G)$ via $\sigma = (1, \dots, 1)$. It has been established that the Coulomb branch of $T^\rho(G)$, which is equivalent to the Higgs branch of $T_\rho(\widehat{G})$ as an algebraic variety, is a nilpotent orbit closure

$$\mathcal{M}_C(T^\rho(G)) \cong \mathcal{M}_H(T_\rho(\widehat{G})) \cong \overline{\mathcal{O}}_{\rho^\vee} \,. \tag{2.4}$$

Here, we need a map $\vee : \rho \mapsto \rho^\vee$ that takes partitions of G to partitions of the GNO dual $\widehat{G}$. Such a map is known [8, 42–44] and is named *Barbasch-Vogan* map, see also [6, sections 4.3-4.4]. For $G = \mathrm{SU}(n)$, one simply has $\rho^\vee = \rho^T$; whereas the other classical groups have slightly more involved prescriptions. Since we will be dealing with unitary quiver realisations of the BCD-type $T^\rho(G)$, the details of the Barbasch-Vogan map are not utterly important and we refer to [8, 17] for explicit expositions.

A-type. For A-type nilpotent orbits, the Coulomb branch quivers (as well as the Higgs branch quivers) are well-behaved and exhaust all possible nilpotent orbits of type A as their moduli space. In particular on the Coulomb branch side, the quiver for A_n orbits

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have exactly n unitary gauge nodes, which allows to compare not only the dimension of the Coulomb branch, but also the full refinement of the Hilbert series, which enables for the decomposition into irreducible representations of $\mathrm{SU}(n+1)$.

BCD-type. The Higgs branch quivers for BCD-type nilpotent orbit closures are built from alternating orthogonal-symplectic gauge nodes and their constructions exhausts all possible nilpotent orbits. However, the Coulomb branch constructions are problematic for a number of reasons. Many of the theories with orthogonal and symplectic gauge nodes are *bad* in the sense of [2]. In other words, the monopole formula defined by Lagrangian data is ill-defined and divergent. Computations of Coulomb branches with non-unitary gauge groups yield the correct unrefined Hilbert series of the claimed orbit closure, but fail to reproduce the fully refined Hilbert series. Fortunately, a unitary quiver construction for near to minimal BCD-type nilpotent orbit closures (of characteristic height two) has been presented in [3] by means of flavoured finite-type Dynkin diagrams. We will focus on these unitary realisations for BCD-type nilpotent orbits.

The quiver gauge theories for D_n have first been computed in [45]; the same quivers appear in [46] in the study of Slodowy slices of B and D-type in the vicinity of the maximal (regular) nilpotent orbit of the respective algebra.

Exceptional types. It is notoriously difficult to obtain exceptional nilpotent orbit closures from standard gauge or string theory constructions. Higgs branch constructions are not available simply because exceptional groups do not act via matrices on a fundamental vector space. While Coulomb branch constructions for minimal orbits are known [10, 47, 48] for some time, constructions for near to minimal nilpotent orbit closures have only been proposed very recently in [4]. This unitary quiver construction again employs flavoured Dynkin diagrams and is limited to the lower dimensional orbits.

Hilbert series. The Hilbert series for the $T_\rho(G)$ theories have been presented in [13, 14]; while the Hilbert series of the general class $T^\rho_\sigma(G)$ has been studied in [17]. It is worthwhile noting that the closures of minimal nilpotent orbits correspond to reduced single instanton moduli spaces. The Hilbert series of these have first been computed in [49] and many other constructions are known [10, 48, 50]. The Hilbert series and HWG for nilpotent orbit closures of classical and exceptional groups have been systematically studied in [3, 4]. Special attention to the distinction between $\mathrm{SO}(N)$ and $\mathrm{O}(N)$ gauge groups in Coulomb branch realisations of $\mathfrak{so}(n)$ nilpotent orbit has been given in [51].

In view of the form of the resolutions (2.1), the HWG for $T^*(G/P)$ have been evaluated in [19] and found to agree with the Coulomb branch nilpotent orbit results.

2.5 Monopole formula with background charges

To study the resolutions of Coulomb branches, one turns on non-trivial real mass parameters. As these transform in the adjoint of the flavour symmetry, the inclusion of real mass parameters can be realised via background charges (fluxes) in the monopole formula [13, 18, 52]. Suppose we are given a 3-dimensional $\mathcal{N}=4$ gauge theory with gauge group $\mathcal{G}$. The GNO dual group is denoted by $\widehat{\mathcal{G}}$, the Weyl group for $\mathcal{G}$ (and $\widehat{\mathcal{G}}$) is $\mathcal{W}$, and

Φ^+ denotes the set of positive roots α of Lie($\mathcal{G}$). Moreover, the matter content transforms in a representation $\oplus_I n_I \mathcal{R}_I$ of $\mathcal{G}$ and a representation $\mathcal{R}_F$ of the flavour symmetry G_F. The weight vectors of $\mathcal{R}_I$ are denoted by ρ, and the weights of $\mathcal{R}_F$ by $\tilde{\rho}$.

Associated to the gauge group $\mathcal{G}$ are (dynamical) bare monopole operators, which are uniquely labelled by lattice points in the GNO weight lattice $\Gamma_{\widehat{\mathcal{G}}}$ up to gauge equivalence [53]. Similarly, there exist (background) monopole operators associated to the flavour symmetry group G_F, which are uniquely labelled by the GNO lattice of G_F (again, up to equivalence).

The monopole charge $q \in \Gamma_{\widehat{\mathcal{G}}}$ breaks the gauge symmetry via adjoint Higgs mechanism to $H_q = \mathrm{Stab}_{\mathcal{G}}(q) \subset \mathcal{G}$. The resulting residual gauge theory may admit further gauge invariant chiral operators that can take non-trivial vacuum expectation values, which are accounted for by the Hilbert series $P_{\mathcal{G}}(t^2, m)$ of the residual gauge symmetry [10]. The combination of non-trivial monopole background and VEVs in the residual gauge symmetry leads to dressed monopole operators.

Thus, we are ready to recall the Coulomb branch Hilbert series[4] in the presence of background charges:

$$\mathrm{HS}_p(t^2, z) = \sum_{q \in \Gamma_{\widehat{\mathcal{G}}}/\mathcal{W}} P_{\mathcal{G}}(t^2, q) \cdot t^{2\Delta(q,p)} \cdot z^{J(q)} \tag{2.5}$$

where

$$\Delta(q,p) = \frac{1}{2} \sum_I \sum_{\rho_I \in \mathcal{R}_I} \sum_{\tilde{\rho} \in \mathcal{R}_F} |\rho(q) + \tilde{\rho}(p)| - \sum_{\alpha \in \Phi^+} |\alpha(q)| \tag{2.6}$$

is the conformal dimension of a monopole operator with charges (q, p), see [2, 53–55]. The dressing factors are given by $P_{\mathcal{G}}(t^2, q) = \prod_{i=1}^{\mathrm{rk}\mathcal{G}} 1/(1 - t^{2d_i})$ with d_i are degrees of the Casimir invariants of H_q, see [10]. Note that we can restrict the background charge to $p \in \Gamma_{\widehat{G}_F}/\mathcal{W}_F$. In addition, we have chosen to account for the topological symmetries $G_J^{\mathrm{UV}} = \mathrm{U}(1)^{\#(\mathrm{U}(1) \text{ in } \mathcal{G})}$ by fugacities $z \equiv (z_i)$ and their charge $J(z)$. The map J is a linear projection map from the GNO weight lattice to the Cartan subalgebra of the flavour symmetry. Similarly to the discussion of [13, equation (3.8)], one can remove an extra overall topological U(1). For a set of flavour nodes labelled by N_i, the physical flavour symmetry is $(\prod_i \mathrm{U}(N_i))/\mathrm{U}(1)$ rather than $(\prod_i \mathrm{U}(N_i))$.

Hilbert series generating function. The monopole formula, with or without background charges, presents a computational challenge due to the step-wise linear behaviour of $\Delta(q, p)$. In earlier works [56, 57], we have introduced a method to systematise and partly overcome these complications by restricting to the domains of linearity of Δ. In the absence of background charges, this procedure naturally leads to affine monoids organised by a fan.

The inclusion of background charges p leads to non-central hyperplanes

$$H_{\rho,\tilde{\rho}}(p) = \left\{ q \in \mathfrak{t} \,\middle|\, \rho(q) + \tilde{\rho}(p) = 0 \right\} \tag{2.7}$$

[4] In order to work with an integer grading, we choose the R-charge of the bare BPS monopole operators to be counted by t^2 instead of t.

and corresponding closed half-spaces $H^{\pm}_{\rho,\tilde{\rho}}(p)$. Here, $\mathfrak{t}$ denotes a Cartan subalgebra of $\mathfrak{g}$. To resolve (2.6), one would intersect all possible half-spaces. Contrary to the $p \equiv 0$ case, the intersection is not necessarily a polyhedral cone, but generically a polyhedron. Although there exists a mathematical notion for the Hilbert series for the intersection of a polyhedron with a lattice [58], we choose to circumvent the resulting problem by computing the generating function for (2.5). Considering

$$\mathcal{F}(t^2, z; y) = \sum_{p \in \Gamma_{\widehat{G}_F}/\mathcal{W}_F} y^p \cdot \mathrm{HS}_p(t^2, z)\,, \tag{2.8}$$

we realise that all the techniques of [56, 57] are straightforwardly applicable. Hence, we use this approach to compute $\mathrm{HS}_p(t^2, z)$ from $\mathcal{F}(t^2, z; y)$.

Highest weight generating function. Having computed $\mathrm{HS}_p(t^2, z)$, the result might not be too illuminating. Fortunately, we can project the Hilbert series onto the *Highest Weight Generating function* (HWG) [59]. To summarise the essentials, the considered unitary quiver theories all have an enhanced infrared global symmetry G_J^{IR} of type $ABCDE$, which is counted in the refined HS by fugacities z_i, $i = 1, \dots, r$ with $r = \mathrm{rk}(G_J^{\mathrm{IR}})$. One transforms the HS into a character expansion of $\mathfrak{g}_J = \mathrm{Lie}(G_J^{\mathrm{IR}})$, via mapping the z_i to new fugacities x_i using the Cartan matrix of $\mathfrak{g}_J$, see appendix A. Hence,

$$\mathrm{HS}_p(t^2, z) = \sum_{n \in \mathbb{N}} f_n(z_i; p) t^n \quad \xrightarrow[\text{matrix}]{\text{Cartan}} \quad \mathrm{HS}_p(t^2, x) = \sum_{n \in \mathbb{N}} \tilde{f}_n(x_i; p) t^n \tag{2.9}$$

and each $\tilde{f}_n(x_i; p)$ can be decomposed into a finite sum of G_J^{IR}-characters $\chi_{[n_1,\dots,n_r]}(x_i)$. Next, one replaces each character by a monomial in highest weight fugacities μ_i, $i = 1, \dots, r$

$$\chi_{[n_1,\dots,n_r]}(x_i) \quad \mapsto \quad \prod_{i=1}^{r} \mu_i^{n_i} \qquad \text{with} \quad n_i \in \mathbb{N} \tag{2.10}$$

such that the HS is transformed into a HWG

$$\mathrm{HWG}_p(t^2, \mu) = \sum_{k_1,\dots,k_r,n \in \mathbb{N}} g_{k_1,\dots,k_r;n}(p)\ \mu_1^{k_1} \cdot \ldots \cdot \mu_r^{k_r} t^n\,. \tag{2.11}$$

For a detailed introduction and the necessary computational tools, we refer to [59]. It is an empirical observation from [3, 4] that the Hilbert series for nilpotent orbit closures of (characteristic) height two have a simple HWG. As we will see below, this is also true for the Coulomb branches in the presence of background charges.

3 A-type

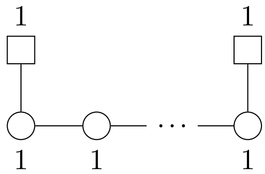
We start by considering the nilpotent orbits of A-type of height two realised as a Coulomb branch. To be specific, consider orbits $\mathcal{O}_\rho$ of A_n with partitions $\rho = (2^k, 1^{n+1-2k})$ for $2 \leq 2k \leq n+1$, such that $\mathrm{ht}(\mathcal{O}_\rho) = 2$. The relevant Coulomb branch quiver gauge theories have been known for some time [2] and the HWG for the singular case have been computed in [3] to read

$$\mathrm{HWG}^{(2^k,1^{n+1-2k})}(t^2) = \mathrm{PE}\left[\sum_{i=1}^{k} \mu_i \mu_{n+1-i} t^{2i}\right] . \tag{3.1}$$

We computed the Hilbert series and HWG in the presence of background charges to study the resolutions of the (closures) of the A_n nilpotent orbits for $n \leq 5$. Moreover, due to [35, Corollary 3.16] we know that all height two nilpotent orbits admit a symplectic resolution (with a suitable polarisation). Now, we need to compare this fact to the Coulomb branch computations, for which we summarise the results in table 3.

Minimal nilpotent orbit. Before considering the generic case, we elaborate on Coulomb branch quivers for the A-type minimal nilpotent closure for two reasons: firstly, to exemplify the calculations and explain the conclusions drawn from it. Secondly, to highlight the special geometry of these abelian theories, which have $\dim_{\mathbb{H}} = n$ Coulomb branches with a $(\mathbb{C}^\times)^n$-action.

From the quiver representation with n U(1) gauge nodes we read off the conformal dimension



$$\rightarrow \quad \Delta(q,p) = \frac{1}{2}\left(\sum_{i=1}^{n-1} |q_i - q_{i+1}| + |q_1 - p_1| + |q_n - p_n|\right) , \tag{3.2}$$

with magnetic charges $q_i \in \mathbb{Z}$, for $i = 1, \ldots, n$, and fluxes $p_1, p_n \in \mathbb{Z}$. Without loss of generality, we consider the case $p_1 \geq p_N$. In the spirit of [56, 57], we understand the absolute values in $\Delta(q,p)$ as defining hyperplanes in $\mathbb{R}^n$. Their intersection gives rise to bounded regions, i.e. polytopes, as well as unbounded regions, i.e. polyhedra. Since any polyhedron can be decomposed as Minkowski sum of a polytope and a polyhedral cone, we only consider the polytopes because we elaborated on how to deal with polyhedral cones in [56, 57]. Let us focus on the maximal dimensional polytope appearing in the summation range $q_i \in \mathbb{Z}$, $i = 1, \ldots, n$.

Considering for a moment $\vec{q} \equiv (q_i) \in \mathbb{R}^n$, then the polytope $\mathcal{P}_z$ is defined by the intersection of the following half-spaces

$$P_z = \{\vec{q} \in \mathbb{R}^n \,|\, q_1 - p_1 \leq 0\} \cap \{\vec{q} \in \mathbb{R}^n \,|\, q_n - p_n \geq 0\} \cap \bigcap_{i=1}^{n-1} \{\vec{q} \in \mathbb{R}^n \,|\, q_i - q_{i+1} \geq 0\} \subset \mathbb{R}^n \tag{3.3}$$

and can equivalently be characterised by its vertices as follows:

$$\mathcal{P}_z = \mathrm{Conv}\left\{\begin{matrix} (p_n, p_n, \ldots, p_n, p_n) \\ (p_1, p_n, \ldots, p_n, p_n) \\ \ldots \\ (p_1, p_1, \ldots, p_1, p_n) \\ (p_1, p_1, \ldots, p_1, p_1) \end{matrix}\right\} \subset \mathbb{R}^n . \tag{3.4}$$

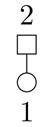
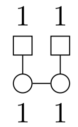
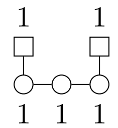
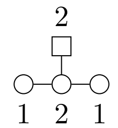
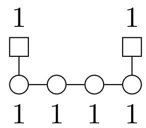
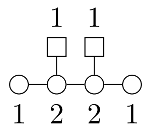
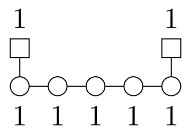
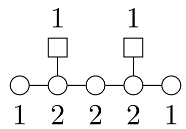
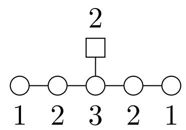
ρ	$\dim_{\mathbb{C}}$	quiver	HWG with flux
(2)	2		$x_1^{p_1+p_2}(\mu_1 t)^{p_1-p_2}\ \mathrm{PE}[\mu_1^2 t^2]$
$(2,1)$	4		$\begin{cases} x_1^{p_1}x_2^{p_2}(\mu_2 t)^{p_1-p_2}\ \mathrm{PE}[\mu_1\mu_2 t^2] & , p_1 \geq p_2 \\ x_1^{p_1}x_2^{p_2}(\mu_1 t)^{p_2-p_1}\ \mathrm{PE}[\mu_1\mu_2 t^2] & , p_1 \leq p_2 \end{cases}$
$(2,1^2)$	6		$\begin{cases} x_1^{p_1}x_3^{p_3}(\mu_3 t)^{p_1-p_3}\ \mathrm{PE}[\mu_1\mu_3 t^2] & , p_1 \geq p_3 \\ x_1^{p_1}x_3^{p_3}(\mu_1 t)^{p_3-p_1}\ \mathrm{PE}[\mu_1\mu_3 t^2] & , p_1 \leq p_3 \end{cases}$
(2^2)	8		$x_2^{p_1+p_2}(\mu_2 t^2)^{p_1-p_2}\ \mathrm{PE}[\mu_1\mu_3 t^2 + \mu_2^2 t^4]$
$(2,1^3)$	8		$\begin{cases} x_1^{p_1}x_4^{p_4}(\mu_4 t)^{p_1-p_4}\ \mathrm{PE}[\mu_1\mu_4 t^2] & , p_1 \geq p_4 \\ x_1^{p_1}x_4^{p_4}(\mu_1 t)^{p_4-p_1}\ \mathrm{PE}[\mu_1\mu_4 t^2] & , p_1 \leq p_4 \end{cases}$
$(2^2,1)$	12		$\begin{cases} x_2^{p_2}x_3^{p_3}(\mu_3 t^2)^{p_2-p_3}\ \mathrm{PE}[\mu_1\mu_4 t^2 + \mu_2\mu_3 t^4] & , p_2 \geq p_3 \\ x_2^{p_2}x_3^{p_3}(\mu_2 t^2)^{p_3-p_2}\ \mathrm{PE}[\mu_1\mu_4 t^2 + \mu_2\mu_3 t^4] & , p_2 \leq p_3 \end{cases}$
$(2,1^4)$	10		$\begin{cases} x_1^{p_1}x_5^{p_5}(\mu_5 t)^{p_1-p_5}\ \mathrm{PE}[\mu_1\mu_5 t^2] & , p_1 \geq p_5 \\ x_1^{p_1}x_5^{p_5}(\mu_1 t)^{p_5-p_1}\ \mathrm{PE}[\mu_1\mu_5 t^2] & , p_1 \leq p_5 \end{cases}$
$(2^2,1^2)$	16		$\begin{cases} x_2^{p_2}x_4^{p_4}(\mu_4 t^2)^{p_2-p_4}\ \mathrm{PE}[\mu_1\mu_4 t^2 + \mu_2\mu_3 t^4] & , p_2 \geq p_4 \\ x_2^{p_2}x_4^{p_4}(\mu_2 t^2)^{p_4-p_2}\ \mathrm{PE}[\mu_1\mu_4 t^2 + \mu_2\mu_3 t^4] & , p_2 \leq p_4 \end{cases}$
(2^3)	18		$x_3^{p_1+p_2}(\mu_3 t^3)^{p_1-p_2}\mathrm{PE}[\mu_1\mu_5 t^2 + \mu_2\mu_4 t^4 + \mu_3^2 t^6]$

Table 3. Coulomb branch quiver gauge theories for A-type algebras: gauge theories $T^{\rho^T}[\mathrm{SU}(n+1)]$ as Coulomb branch realisations of the (closures of the) nilpotent orbits $\overline{\mathcal{O}}_\rho$ of A_n, for $n = 1,2,3,4,5$. The unphysical U(1) in G_F can be eliminated in the HWG by imposing that the sum of fluxes vanishes. In more detail, the U(1) could be counted by an auxiliary fugacity z_0 and one can impose $z_0^{p_k+p_{n+1-k}}\prod_{i=1}^n z_i^{r_i} = 1$, where r_i are the ranks of the n gauge groups. Setting $z_0 = 1$ and converting root space to weight space fugacities, one obtains the condition $x_k x_{n+1-k} = 1$.

From the refined monopole formula (2.5), we see that $\mathcal{P}_z$ is a polytope in the root lattice, spanned by $z_1, \dots, z_n$. Next, we utilise the Cartan matrix of A_n to map $\mathcal{P}_z$ via (A.1) into a polytope $\mathcal{P}_x$ in the weight lattice, spanned by $x_1, \dots, x_n$. This is exactly the same transformation as in (2.9). The polytope $\mathcal{P}_x$ is again defined by its vertices:

$$\mathcal{P}_x = \text{Conv} \left\{ \begin{matrix} (p_n, 0, \dots, 0, p_n) \\ (2p_1 - p_n, p_n - p_1, \dots, 0, p_n) \\ \dots \\ (p_1, 0, \dots, p_1 - p_n, 2p_n - p_1) \\ (p_1, 0, \dots, 0, p_1) \end{matrix} \right\} \subset \mathbb{R}^n \,. \tag{3.5}$$

Splitting off an off-set vector $(p_1, 0, \dots, 0, p_n)$ and realising a dilation factor $p_1 - p_n \geq 0$, we rewrite the polytope $\mathcal{P}_x$ as

$$\mathcal{P}_x = (p_1, 0, \dots, 0, p_n) \ + \ (p_1 - p_n) \times \text{Conv}(S) \tag{3.6a}$$

$$S = \left\{ \begin{matrix} (-1, 0, \dots, 0, 0) \\ (1, -1, \dots, 0, 0) \\ \dots \\ (0, 0, \dots, 1, -1) \\ (0, 0, \dots, 0, 1) \end{matrix} \right\} \,. \tag{3.6b}$$

Taking the intersection with the GNO weight lattice $\mathbb{Z}^n$, we observe that $\mathcal{P}_x \cap \mathbb{Z}^n$ agrees with the weight vectors of the $\mathrm{SU}(n+1)$ representation $[0, 0, \dots, 0, p_1 - p_n]$ shifted by an off-set $(p_1, 0, \dots, 0, p_n)$. This follows because the set $S \cap \mathbb{Z}^n$ agrees with the weights of $[0, 0, \dots, 0, 1]$, and $p_1 - p_n$ yields the dilation to $[0, 0, \dots, 0, p_1 - p_n]$. Consequently, the contribution of $\mathcal{P}_x \cap \mathbb{Z}^n$ to the HWG is

$$\mathrm{HWG}_{(p_1, p_n)}(\mathcal{P}_x \cap \mathbb{Z}^n) = x_1^{p_1} x_n^{p_n} \ (\mu_n t)^{p_1 - p_n} \,. \tag{3.7}$$

Comparing to the full HWG for $(2, 1^{n-1})$, as shown in table 3 or below in (3.10), we see that (3.7) describes the ratio of the HWG with and without background fluxes. The case $p_n \geq p_1$ produces the representation $[p_n - p_1, 0, \dots, 0]$ instead, such that the HWG is changed appropriately.

More geometrically, we recognise $S \cap \mathbb{Z}^n$ as standard simplex in $\mathbb{R}^n$. Since abelian 3-dimensional $\mathcal{N} = 4$ theories are known to have hyper-toric Higgs and Coulomb branches, we might be tempted to take the standard simplex as indicator for a $\mathbb{C}P^n$. In fact, we would understand the $(p_1 - p_n)$-dilated simplex as giving the $T^*\mathbb{C}P^n$, where the flux $(p_1 - p_n)$ determines the size of the $\mathbb{C}P^n$. Similarly to SQED with N flavours [13, 18], we can define operators on the vertices of $\mathcal{P}_z \cap \mathbb{Z}^n$ which realise the correct transition functions between the affine patches of $\mathbb{C}P^n$. Let us illustrate this for $\mathbb{C}P^2$ as follows: for $n = 2$ the polyhedron $\mathcal{P}_z$ is defined by the three edges (p_1, p_1), (p_1, p_2), and (p_2, p_2). Define the following operators/coordinates, see also figure 1:

$$\mathcal{U}_1 : \qquad V_{(p_2+a+b, p_2+b)} = V_{(p_2, p_2)} X^a Z^b \tag{3.8a}$$

$$\mathcal{U}_2 : \qquad V_{(p_1-c, p_2+d)} = V_{(p_1, p_2)} U^c V^d \tag{3.8b}$$

$$\mathcal{U}_3 : \qquad V_{(p_1-e, p_1-e-f)} = V_{(p_1, p_1)} Y^e W^f \tag{3.8c}$$

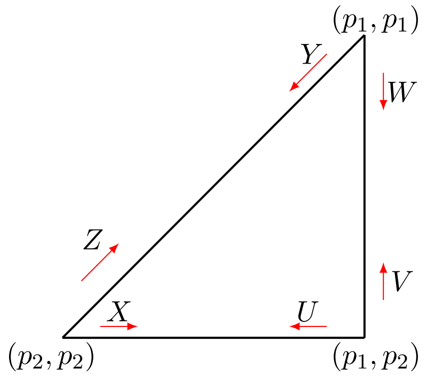


Figure 1. The polytope $\mathcal{P}_z$ arising in the monopole formula for the minimal nilpotent orbit closure of A_2.

On the overlap of, say, $\mathcal{U}_1 \cap \mathcal{U}_2$ we find

$$\begin{aligned}
& V_{(p_2+a+b,p_2+b)} = V_{(p_1-(p_1-p_2-a-b),p_2+b)} \quad \forall a,b \\
\Leftrightarrow \quad & V_{(p_2,p_2)} X^a Z^b = V_{(p_1,p_2)} U^{p_1-p_2-a-b} V^b \quad \forall a,b \\
\Leftrightarrow \quad & V_{(p_2,p_2)} (XU)^a (ZU)^b = V_{(p_1,p_2)} U^{p_1-p_2} V^b \quad \forall a,b \\
\Leftrightarrow \quad & XU = 1\,, \qquad ZU = V\,,
\end{aligned} \tag{3.9}$$

which are precisely the transition functions of $\mathbb{C}P^2$.

In view of the known resolutions $\pi : T^*(\mathbb{C}P^n) \to \overline{\mathcal{O}}_{\text{min}}$, it is suggestive to interpret the contribution of the bounded summation region as giving $T^*\mathbb{C}P^n$, where $\mathbb{C}P^n$ is of size $(p_1 - p_n)$. The size can also be seen from the factor $t^{p_1-p_n}$ in (3.7). This follows, because for all $p_1 - p_n > 0$ the geometry of the resolved space is $T^*(\mathbb{C}P^n)$, but the size of the $\mathbb{C}P^n$ is not fixed yet. The entire argument becomes even more compelling by recalling that the two resolutions for $p_1 \geq p_n$ and $p_1 \leq p_n$ are manifestations of the Mukai flop of type A, see (2.2).

General height two case. To begin with, we observe that all Coulomb branch quiver gauge theories allow for non-trivial resolution parameters, as the flavour symmetry groups are either $(\mathrm{U}(1)\times\mathrm{U}(1))/\mathrm{U}(1)$ or $\mathrm{SU}(2)$. From the examples computed, we can even conjecture the Coulomb branch Hilbert series in the presence of background charges for all $\overline{\mathcal{O}}_{(2^k,1^{n+1-2k})}$ of $\mathfrak{sl}(n+1)$. We propose

$$\mathrm{HWG}^{(2^k,1^{n+1-2k})}_{(p_k,p_{n+1-k})}(t^2) = \begin{cases} x_k^{p_k} x_{n+1-k}^{p_{n+1-k}} \left(\mu_{n+1-k} t^k\right)^{p_k - p_{n+1-k}} \mathrm{PE}\left[\sum_{i=1}^k \mu_i \mu_{n+1-i} t^{2i}\right], & p_k \geq p_{n+1-k} \\ x_k^{p_k} x_{n+1-k}^{p_{n+1-k}} \left(\mu_k t^k\right)^{p_{n+1-k} - p_k} \mathrm{PE}\left[\sum_{i=1}^k \mu_i \mu_{n+1-i} t^{2i}\right], & p_k < p_{n+1-k} \end{cases}. \tag{3.10}$$

To remove the overall U(1) shift symmetry in G_F, one simply has to impose $p_k + p_{n+1-k} = 0$. This reduces the problem to one effective resolution parameter $\sim \pm(p_k - p_{n+1-k})$.

Inspecting the expression (3.10), we observe two prominent features: firstly, the HWG of the resolved space factors into the HWG (3.1) of the singular space times a prefactor. Secondly, depending on the ordering of p_k and p_{n+1-k} the HWG becomes case dependent. Now, we aim to explain these observations.

As elaborated earlier, $\overline{\mathcal{O}}_\rho$ is resolved via $\pi_\rho : T^*(G/P_\rho) \to \overline{\mathcal{O}}_\rho$ with $\pi_\rho^{-1}(0) \cong G/P_\rho$. For the height two partitions $\rho = (2^k, 1^{n+1-2k})$, the relevant coset spaces are the Grassmann manifolds

$$G_{m,k} \cong \frac{\mathrm{SU}(m+k)}{\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(k))}\,, \tag{3.11}$$

which enjoy the isomorphism $G_{m,k} \cong G_{k,m}$. Moreover, for $k=1$ one obtains $G_{m,1} \cong \mathbb{C}P^m$. Consequently, the height two orbits of A-type are resolved by cotangent bundles of the Hermitian symmetric spaces $G_{m,k}$.

It is known that these spaces also appear as (semi-simple) coadjoint orbits of the fundamental weights μ_k, $k = 1, \ldots, n$ of A_n. To see this, note that the stabilisers are given by

$$\mathrm{Stab}_{\mathrm{SU}(n+1)}(\mu_k) \cong S(\mathrm{U}(k) \times \mathrm{U}(n+1-k))\,, \tag{3.12}$$

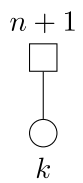
and observe that $\mathrm{Stab}_{\mathrm{SU}(n+1)}(\mu_k) \cong \mathrm{Stab}_{\mathrm{SU}(n+1)}(\mu_{n+1-k})$. Thus, we obtain the semi-simple orbits

$$\mathcal{O}^{\mathrm{ss}}_{\mu_k} = G_{k,n+1-k} \cong G_{n+1-k,k} = \mathcal{O}^{\mathrm{ss}}_{\mu_{n+1-k}}\,. \tag{3.13}$$

Hence, we identify the prefactor μ_k or μ_{n+1-k} in (3.10) as accounting for the holomorphic sections on the cotangent bundle $T^*G_{k,n+1-k}$ over the exceptional fibre $\cong \mathcal{O}^{\mathrm{ss}}_{\mu_k} = G_{k,n+1-k}$.

Next, the existence of two resolutions for $2k < n+1$ is the manifestation of the Mukai flop of type A. In other words, each of the height two nilpotent orbits has two symplectic resolutions, which are related by a Mukai flop (2.2). The observation that the HWG changes depending on the relative sign of $p_k - p_{n+1-k}$ is consistent with this statement, as the prefactor change from μ_k to μ_{n+1-k} indicates the Mukai flop from $G_{k,n+1-k}$ to $G_{n+1-k,k}$. Consistently, we observe for $2k = n+1$, i.e. for the examples $\rho = (2)$ of A_1, $\rho = (2^2)$ of A_3, and $\rho = (2^3)$ of A_5, etc. that only one resolution exists, because the two potential resolutions are isomorphic as discussed below (2.2). (In this case, it is implicitly understood that the fluxes in (3.10) are relabelled to p_1, p_2 satisfying $p_1 \geq p_2$.)

Higgs branch. To complete the aforementioned reasoning, consider the mirror quiver of partition $\rho^T = (n+1-k, k)$: i.e. SQCD with U(k) gauge group and $n+1$ flavours



(3.14)

such that the Higgs branch becomes $\overline{\mathcal{O}}_{(2^k,1^{n+1-2k})}$. To see the cotangent bundle of Grassmann manifolds, start with $k=1$ and recall the $\mathcal{N}=2$ field content of SQED: the $(n+1)$

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$\mathcal{N}=4$ hypermultiplets split into $\mathcal{N}=2$ hypermultiplets X_i and Y_i, with $i=1,\ldots,n+1$ such that the U(1) charges for (X_i,Y_i) are $(1,-1)$. The F and D-term equations are understood as complex and real moment maps

$$\mu_{\mathbb{C}} = \sum_{i=1}^{n+1} X_i Y_i \,, \qquad \mu_{\mathbb{R}} = \sum_{i=1}^{n+1} |X_i|^2 - \sum_{i=1}^{n+1} |Y_i|^2 \tag{3.15}$$

such that the Higgs branch is the hyper-Kähler quotient

$$\mathcal{M}_H = (\mu_{\mathbb{R}}=0, \mu_{\mathbb{C}}=0)/\mathrm{U}(1)\,. \tag{3.16}$$

Turing on a non-trivial real FI parameter $\xi_{\mathbb{R}}$ leads to $\mu_{\mathbb{R}}=\xi_{\mathbb{R}}$, which enforces either $X_i=0$ or $Y_i=0$ for all $i=1,\ldots,n+1$, depending on the sign of $\xi_{\mathbb{R}}$. To recognise the $\mathbb{C}P^n$ base, recall that the complexified U(1) gauge group action identifies $(X_1,\ldots,X_{n+1}) \sim (\lambda X_1,\ldots,\lambda X_{n+1})$ for $\lambda \in \mathbb{C}^\times = \mathrm{U}(1)^{\mathbb{C}}$ and $\xi_{\mathbb{R}}>0$. Moreover, the two different resolutions for $\xi_{\mathbb{R}} \gtrless 0$ are a manifestation of the basic A-type Mukai flop. Hence, $(\mu_{\mathbb{R}}=\xi_{\mathbb{R}}, \mu_{\mathbb{C}}=0)/\mathrm{U}(1) \cong T^*\mathbb{C}P^n$ describes the resolved Higgs branch as a complex manifold.

Generalising to $k\geq 2$, one repeats the same reasoning for F and D terms and observes that the complexified $\mathrm{U}(k)^{\mathbb{C}} = \mathrm{GL}(k,\mathbb{C})$ gauge transformations identify the Higgs branch coordinates to a Grassmann manifold $G_{k,n+1-k}$ or $G_{n+1-k,k}$, depending on the sign of the real FI parameter. Note that the FI-term measures the size of the Grassmann manifold, see also [15].

4 B-type

The Coulomb branch realisation of B-type nilpotent orbit closures via unitary quiver gauge theories have been constructed in [3]. In terms of partitions, there are the following two families for B_n which are of height two: firstly, $\rho=(2^{2k},1^{2n+1-4k})$ for $4\leq 4k\leq 2n+1$ and, secondly, $\rho=(3,1^{2n-2})$. We provide the computational results for Coulomb branches corresponding to height two orbits of B_n with $n=2,3,4$ in table 4. The remarkable observation is that the HWG with fluxes factors neatly into a prefactor times the HWG of the singular case, computed in [3].

We recall that [30] provides information on the existence and form of the resolution of the nilpotent orbits for B_n with $n=2,3,4$. For larger ranks, one can consult [35, Proposition 3.19]: let ρ be a B-type partition of $2n+1$, then there exist a symplectic resolution of $\overline{\mathcal{O}}_\rho$ (and suitable polarisation) if and only if there exist an odd number $q\geq 0$ such that the first q parts of ρ are odd and the other parts are even. Applying this criterion to the two height two families, we find:

(i) $\rho=(2^{2k},1^{2n+1-4k})$: there does not exist a resolution, since $k>0$.

(ii) $\rho=(3,1^{2n-2})$: there exists a resolution for any n, since we can choose $q=2n-1$.

Let us compare this to the monopole formula computations.

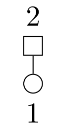
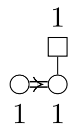
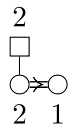
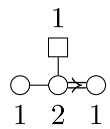
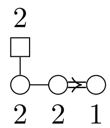
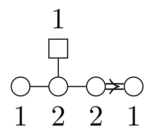
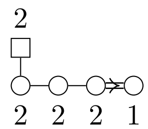
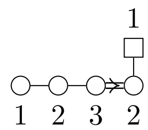
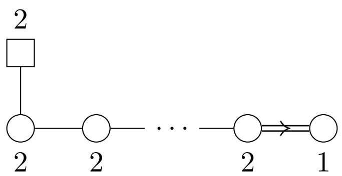
ρ	$\dim_{\mathbb{C}}$	quiver	HWG with flux
(3)	2		$x_1^{p_1+p_2}\,(\mu_1 t)^{p_1-p_2}\;\;\mathrm{PE}\left[\mu_1^2 t^2\right]$
$(2^2,1)$	4		no resolution
$(3,1^2)$	6		$x_1^{p_1+p_2}\,\left(\mu_1 t^2\right)^{p_1-p_2}\cdot\mathrm{PE}\left[\mu_2^2 t^2+\mu_1^2 t^4\right]$
$(2^2,1^3)$	8		no resolution
$(3,1^4)$	10		$x_1^{p_1+p_2}\,\left(\mu_1 t^2\right)^{p_1-p_2}\cdot\mathrm{PE}\left[\mu_2 t^2+\mu_1^2 t^4\right]$
$(2^2,1^5)$	12		no resolution
$(3,1^6)$	14		$x_1^{p_1+p_2}\,\left(\mu_1 t^2\right)^{p_1-p_2}\cdot\mathrm{PE}\left[\mu_2 t^2+\mu_1^2 t^4\right]$
$(2^4,1)$	16		no resolution

Table 4. Coulomb branch quiver gauge theories for B-type algebras: realisations of the (closures of the) nilpotent orbits of height 2 for B_n with $n = 1, 2, 3, 4$. The unphysical $\mathrm{U}(1)$ in G_F can be eliminated as before: introducing an auxiliary fugacity z_0 for $\mathrm{U}(1) \subset G_F$, and imposing $z_0^{\sum_j p_j} \prod_{i=1}^{n} z_i^{r_i} = 1$, with r_i the ranks of the gauge nodes, leads to $x_1 = 1$ for $(3, 1^{2n-2})$ (and $z_0 \equiv 1$). This is morally equivalent to setting the sum of fluxes to zero in the HWG.

Partition $\rho = (2^{2k}, 1^{2n+1-4k})$. For the minimal nilpotent orbit of B_n with partition $(2^2, 1^{2n-3})$, the Coulomb branch does not allow for a resolution parameter as there is only a single U(1) flavour charge. In terms of the monopole formula, a non-trivial U(1) background flux can be absorbed by a simple redefinition of all charges. The non-existence of a resolution for $\overline{\mathcal{O}}_{(2^2,1^{2n-3})}$ is consistent with [30, 35].

We computed only one other member of the family, namely $(2^4, 1)$, for which the monopole formula does not give rise to any resolution parameter. Hence, the constructions are consistent with the mathematical results.

Partition $\rho = (3, 1^{2n-2})$. The Coulomb branch quivers for the next-to-minimal orbit $\overline{\mathcal{O}}_{(3,1^{2n-2})}$ have a U(2) flavour node, thus admit for a non-trivial resolution parameter. Based on the examples computed, we expect that the HWG for the entire family with $n \geq 3$



(4.1)

is given by

$$\mathrm{HWG}^{(3,1^{2n-2})}_{(p_1,p_2)}(t^2) = x_1^{p_1+p_2} \left(\mu_1 t^2\right)^{p_1-p_2} \cdot \mathrm{PE}\left[\mu_2 t^2 + \mu_1^2 t^4\right] . \tag{4.2}$$

To eliminate the overall U(1) factor in G_F, one imposes $p_1 + p_2 = 0$, which reduces the effective flux to $p_1 - p_2 \geq 0$, i.e. an SU(2) background charge corresponding to a single resolution parameter. Again, we can compare the result to the known properties of the resolution

$$\pi_{(3,1^{2n-2})} : T^* \left(\frac{\mathrm{SO}(2n+1)}{\mathrm{SO}(2n-1)\times\mathrm{SO}(2)}\right) \to \overline{\mathcal{O}}_{(3,1^{2n-2})} . \tag{4.3}$$

The symplectic resolution is given by the cotangent bundle of a Hermitian symmetric space. Moreover, the exceptional fibre $\pi^{-1}_{(3,1^{2n-2})}(0) \cong \frac{\mathrm{SO}(2n+1)}{\mathrm{SO}(2n-1)\times\mathrm{SO}(2)}$ is reflected in the prefactor $\propto \mu_1$, as the (semi-simple) coadjoint orbit $\mathcal{O}^{\mathrm{ss}}_{\mu_1}$ of the first fundamental weight μ_1 of B_n is precisely this Hermitian symmetric space. To see this, note that the stabiliser of μ_1 in $\mathrm{SO}(2n+1)$ is $\mathrm{SO}(2n-1) \times \mathrm{SO}(2)$.

5 C-type

In this section, we investigate the resolutions of nilpotent orbits of C-type via background charges in the monopole formula for the Coulomb branch quivers. The construction via unitary quivers as well as the Hilbert series and HWG have been provided in [3]. Restricting to height two, there is exactly one family of partitions for C_n to consider: $\rho = (2^k, 1^{2(n-k)})$ for $1 \leq k \leq n$. We considered all height two cased for C_n with $n = 2, 3, 4$ and summarise the computational results in table 5.

As in the previous case, existence and form of the resolution have been tabulated in [30] for low rank cases. The general criterion has been formulated in [35, Proposition 3.19]: let

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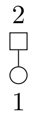
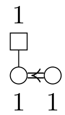
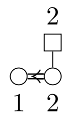
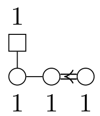
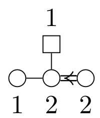
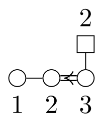
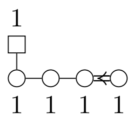
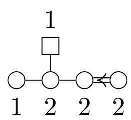
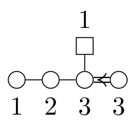
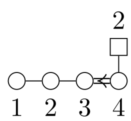
ρ	$\dim_{\mathbb{C}}$	quiver	HWG with flux
(2)	2		$x_1^{p_1+p_2}(\mu_1 t)^{p_1-p_2}\ \mathrm{PE}[\mu_1^2 t^2]$
$(2,1^2)$	4		no resolution
(2^2)	6		$x_2^{p_1+p_2}\left(\mu_2 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^2+\mu_2^2 t^4\right]$
$(2,1^4)$	6		no resolution
$(2^2,1^2)$	10		no resolution
(2^3)	12		$x_3^{p_1+p_2}\left(\mu_3 t^3\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^2+\mu_2^2 t^4+\mu_3^2 t^6\right]$
$(2,1^6)$	8		no resolution
$(2^2,1^4)$	14		no resolution
$(2^3,1^2)$	18		no resolution
(2^4)	20		$x_4^{p_1+p_2}\left(\mu_4 t^4\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^2+\mu_2^2 t^4+\mu_3^2 t^6+\mu_4^2 t^8\right]$

Table 5. Coulomb branch quiver gauge theories for C-type algebras: realisations of the (closures of the) nilpotent orbits of height 2 for C_n, for $n = 1, 2, 3, 4$. To eliminate the unphysical U(1) in G_F in the HWG, one proceeds as before: introducing an auxiliary fugacity z_0 for $\mathrm{U}(1) \subset G_F$, and imposing $z_0^{\sum_j p_j} \prod_{i=1}^n z_i^{r_i} = 1$, with r_i the ranks of the gauge nodes, leads to $x_n = 1$ for (2^n) (and $z_0 \equiv 1$). This is effectively the same as setting the sum of fluxes to zero in the HWG.

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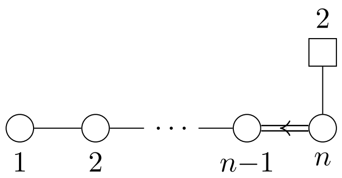
ρ be a C-type partition of $2n$, then there exist a symplectic resolution of $\overline{\mathcal{O}}_\rho$ (and suitable polarisation) if and only if there exists an even number $q \geq 0$ such that the first q parts of ρ are odd and the other parts are even. Inspecting the height two family $\rho = (2^k, 1^{2(n-k)})$ we find: there exists a resolution only for $n = k$, as we then choose $q = 0$. Hence, $\rho = (2^n)$ admits a resolution and $\rho = (2^k, 1^{2(n-k)})$ with $n > k \geq 1$ does not.

Partition $\rho = (2^k, 1^{2(n-k)})$, $n > k \geq 1$. For the minimal nilpotent orbit $\overline{\mathcal{O}}_{(2,1^{2n-2})}$ of C_n, the Coulomb branch quivers do not allow for a resolution parameter, as there is only a single U(1) flavour node present. Next, the orbits of partition $(2^2, 1^{2n-4})$, $n \geq 3$ do not admit a resolution. The Coulomb branch quiver construction are consistent with this. In addition, the quivers corresponding to the partition $(2^3, 1^{2n-6})$ do not give rise to any resolution, because the only available flavour is a U(1) node. Hence, the considered examples do agree with [30, 35].

Partition $\rho = (2^n)$. Lastly, the orbits of partition (2^n) for C_n exhibit a U(2) flavour; thus, a non-trivial resolution parameter exists. From the examples computed, we expect that the HWG for the entire family



(5.1)

is given by

$$\mathrm{HWG}^{(2^n)}_{(p_1,p_2)}(t^2) = x_n^{p_1+p_2} (\mu_n t^n)^{p_1-p_2} \cdot \mathrm{PE}\left[\sum_{i=1}^{n} \mu_i^2 t^{2i}\right] . \tag{5.2}$$

In order to eliminate the overall U(1) in G_F, one imposes $p_1 + p_2 = 0$. This reduces the fluxes to a single $p_1 - p_2 \geq 0$ background charge of SU(2), which corresponds to exactly one resolution parameter. Comparing the result to the known resolution

$$\pi_{(2^n)} : T^* \left(\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}\right) \to \overline{\mathcal{O}}_{(2^n)} , \tag{5.3}$$

we observe again the cotangent bundle of a Hermitian symmetric space, see table 2. In addition, the exceptional fibre $\pi^{-1}_{(2^n)}(0) \cong \frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}$ is reflected in the HWG by the prefactor $\propto \mu_n$. To see this, recall that the stabiliser of the n-th fundamental weight μ_n of C_n is given by $\mathrm{U}(n) \cong \mathrm{SU}(n) \times \mathrm{U}(1)$. Hence, the (semi-simple) coadjoint orbit $\mathcal{O}^{\mathrm{ss}}_{\mu_n}$ through λ_n is isomorphic to the Hermitian symmetric space $\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}$.

Remarks. As a consistency check, one observes that the accidental isomorphism $B_2 \cong C_2$ is respected by the Coulomb branch computations. For instance, the B-type $(3, 1^2)$ of table 4 agrees with C-type (2^2) of table 5 upon identification of fugacity and weight labels. Similarly, the isomorphism $A_1 \cong B_1 \cong C_1$ is manifest in the results.

6 D-type

The last classical case to consider is the nilpotent orbits of D-type. The Coulomb branch construction of the unitary quiver as well as the Hilbert series and HWG has been given in [3]. Here, we analyse the resolutions via background charges in the monopole formula. Restricting ourselves to height two, there are only two families of partitions for D_n to consider: firstly, $\rho = (2^{2k}, 1^{2n-4k})$ for $2 \leq 2k \leq n$ and, secondly, $\rho = (3, 1^{2n-3})$ for $n \geq 2$. The details of the height two cases considered for D_n with $n = 3, 4, 5$ are provided in table 6.

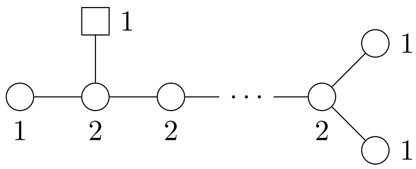
The existence and form of polarisations for low rank D-type nilpotent orbits is tabulated in [30]. For the general statement, we refer to [35, Proposition 3.20]: let ρ be a D-type partition of $2n$, then there exist a symplectic resolution of $\overline{\mathcal{O}}_\rho$ (and suitable polarisation) if and only if either there exists an even number $q \neq 2$ such that the first q parts of ρ are odd and the other parts are even, or there exist exactly 2 odd parts which are at position $2k-1$ and $2k$ in ρ for some k. Inspecting the two height two families, we find the following:

(i) $\rho = (2^{2k}, 1^{2n-4k})$: there exists a resolution either for $2k = n-1$, i.e. for $(2^{n-1}, 1^2)$ of D_n with $n =$ odd, or for $2k = n$, i.e. for (2^n) of D_n with $n =$ even, which is a very even partition. For all other choices of k there does not exist a symplectic resolution.

(ii) $\rho = (3, 1^{2n-3})$: there exists a resolution for any n.

Let us compare this to the monopole formula computations.

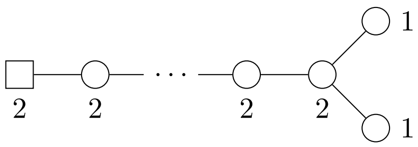
Partition $\rho = (2^{2k}, 1^{2n-4k})$, $n-1 > 2k \geq 2$. To begin with, consider the minimal nilpotent orbit $\mathcal{O}_{(2^2, 1^{2n-4})}$ of D_n. The Coulomb branch quiver



(6.1)

does not allow for any non-trivial resolution parameter. The monopole formula is consistent with the results of [30, 35].

Partition $\rho = (3, 1^{2n-3})$. Next, we consider $\overline{\mathcal{O}}_{(3,1^{2n-3})}$ of D_n, $n \geq 4$, for which the Coulomb branch quiver is the following:



(6.2)

.

Based on the examples computed, we expect that the HWG for the entire family is given by

$$\mathrm{HWG}_{(p_1,p_2)}^{(3,1^{2n-3})}(t^2) = x_1^{p_1+p_2} \left(\mu_1 t^2\right)^{p_1-p_2} \cdot \mathrm{PE}\left[\mu_2 t^2 + \mu_1^2 t^4\right] . \tag{6.3}$$

One can eliminate the overall shift symmetry in the fluxes via the condition $p_1 + p_2 = 0$ such that one obtains a single SU(2) background charge $p_1 - p_2 \geq 0$, which corresponds to

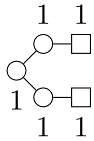
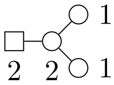
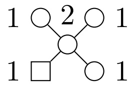
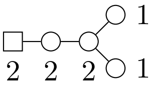
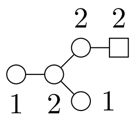
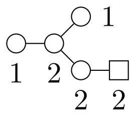
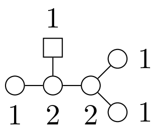
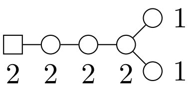
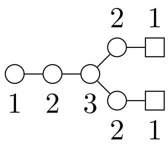
ρ	$\dim_{\mathbb{C}}$	quiver	HWG with flux
$(2^2,1^2)$	6		$\begin{cases} x_2^{p_2}x_3^{p_3}\,(\mu_3 t)^{p_2-p_3}\cdot \mathrm{PE}\left[\mu_2\mu_3 t^2\right] & p_2\geq p_3 \\ x_2^{p_2}x_3^{p_3}\,(\mu_2 t)^{p_3-p_2}\cdot \mathrm{PE}\left[\mu_2\mu_3 t^2\right] & p_2\leq p_3 \end{cases}$
$(3,1^3)$	8		$x_1^{p_1+p_2}\left(\mu_1 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^4+\mu_2\mu_3 t^2\right]$
$(2^2,1^4)$	10		no resolution
$(3,1^5)$	12		$x_1^{p_1+p_2}\left(\mu_1 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^4+\mu_2 t^2\right]$
$(2^4)^I$	12		$x_3^{p_1+p_2}\left(\mu_3 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_3^2 t^4+\mu_2 t^2\right]$
$(2^4)^{II}$	12		$x_4^{p_1+p_2}\left(\mu_4 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_4^2 t^4+\mu_2 t^2\right]$
$(2^2,1^6)$	14		no resolution
$(3,1^7)$	16		$x_1^{p_1+p_2}\left(\mu_1 t^2\right)^{p_1-p_2}\cdot \mathrm{PE}\left[\mu_1^2 t^4+\mu_2 t^2\right]$
$(2^4,1^2)$	20		$\begin{cases} x_4^{p_4}x_5^{p_5}\left(\mu_5 t^2\right)^{p_4-p_5}\cdot \mathrm{PE}\left[\mu_2^2 t^2+\mu_4\mu_5 t^4\right] & p_4\geq p_5 \\ x_4^{p_4}x_5^{p_5}\left(\mu_4 t^2\right)^{p_5-p_4}\cdot \mathrm{PE}\left[\mu_2^2 t^2+\mu_4\mu_5 t^4\right] & p_4\leq p_5 \end{cases}$

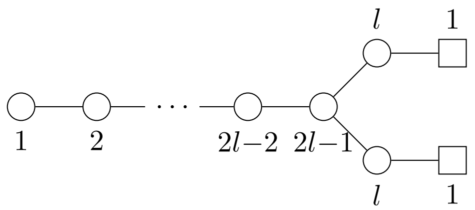
Table 6. Coulomb branch quiver gauge theories for D-type algebras: realisations of the (closures of the) nilpotent orbits of height 2 for D_n with $n=3,4,5$. To eliminate the unphysical U(1) in G_F one repeats the earlier arguments. Introducing an auxiliary fugacity z_0 for $\mathrm{U}(1)\subset G_F$, and imposing $z_0^{\sum_j p_j}\prod_{i=1}^n z_i^{r_i}=1$, with r_i the ranks of the gauge nodes, leads to the following cases: $x_1=1$ for $(3,1^{2n-3})$, $x_{n-1}x_n=1$ for $(2^{n-1},1^2)$ and $n=$ odd, and $x_{n-1}=1$ or $x_n=1$ for (2^n) and $n=$ even (and $z_0\equiv 1$).

one effective resolution parameter. The structure of the results suggest to compare it to the known resolution

$$\pi_{(3,1^{2n-3})} : T^* \left(\frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)\times\mathrm{SO}(2)} \right) \to \overline{\mathcal{O}}_{(3,1^{2n-3})} \tag{6.4}$$

for which the exceptional fibre is $\pi^{-1}_{(3,1^{2n-3})}(0) \cong \frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)\times\mathrm{SO}(2)}$. The prefactor $\propto \mu_1$ indicates this, because the stabiliser of the first fundamental weight μ_1 of D_n is $\mathrm{SO}(2n-2)\times\mathrm{U}(1)$. Hence, the (semi-simple) coadjoint orbit $\mathcal{O}^{\mathrm{ss}}_{\mu_1}$ through μ_1 is isomorphic to the HSS $\frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)\times\mathrm{SO}(2)}$.

Partition $\rho = (2^{n-1}, 1^2)$, n = odd. Consider the orbit closure $\overline{\mathcal{O}}_{(2^{n-1},1^2)}$ of $D_n \equiv D_{2l+1}$ via its Coulomb branch realisation



(6.5)

As apparent from the quiver, the Coulomb branch allows for a resolution parameter and based on the examples computed, we anticipate the HWG to be

$$\mathrm{HWG}^{(2^{n-1},1^2)}_{(p_{2l},p_{2l+1})}(t^2) = \begin{cases} x_{2l}^{p_{2l}} x_{2l+1}^{p_{2l+1}} \left(\mu_{2l+1} t^2\right)^{p_{2l}-p_{2l+1}} \cdot \mathrm{PE}\left[\sum_{i=1}^{l-1} \mu_{2i} t^{2i} + \mu_{2l}\mu_{2l+1} t^{n-1} \right], & p_{2l} \geq p_{2l+1} \\ x_{2l}^{p_{2l}} x_{2l+1}^{p_{2l+1}} \left(\mu_{2l} t^2\right)^{p_{2l}-p_{2l+1}} \cdot \mathrm{PE}\left[\sum_{i=1}^{l-1} \mu_{2i} t^{2i} + \mu_{2l}\mu_{2l+1} t^{n-1} \right], & p_{2l} \leq p_{2l+1} \end{cases} . \tag{6.6}$$

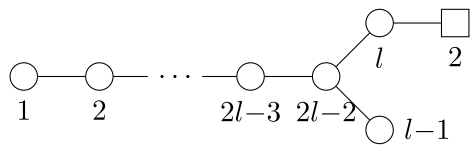
As before, one can eliminate the overall U(1) in G_F by imposing $p_{2l} + p_{2l+1} = 0$, which reduces the system to a single (positive) resolution parameter $\sim \pm(p_{2l} - p_{2l+1})$. Comparing this to the literature [30], the resolution is of the form

$$\pi_{(2^{n-1},1^2)} : T^* \left(\frac{\mathrm{SO}(2n)}{\mathrm{SU}(n)\times\mathrm{U}(1)} \right) \to \overline{\mathcal{O}}_{(2^{n-1},1^2)} , \tag{6.7}$$

with exceptional fibre $\pi^{-1}_{(2^{n-1},1^2)}(0) \cong \frac{\mathrm{SO}(2n)}{\mathrm{SU}(n)\times\mathrm{U}(1)}$. As apparent from the HWG, there are again two cases for the resolution, depending on the relative sign of $p_{2l} - p_{2l+1}$. This is a manifestation of the Mukai flop of D-type, see (2.3).

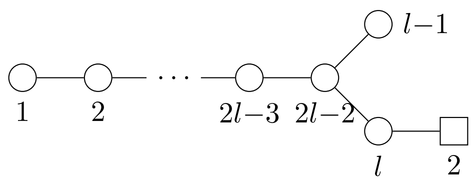
Partition $\rho = (2^n)$, n = even. Consider the orbit closure $\overline{\mathcal{O}}_{2^n}$ of $D_n \equiv D_{2l}$ with the corresponding Coulomb branch quiver

$(2^n)^I$:



, (6.8a)

$(2^n)^{II}$:



. (6.8b)

Note that one obtains two quiver gauge theories, because the very even partition (2^n) corresponds to two nilpotent orbits. The monopole formula admits non-trivial resolution parameters as there is a U(2) flavour node present. Based on the examples computed, we expect the HWG to be

$$(2^n)^I : \qquad \mathrm{HWG}^{(2^n)}_{(p_1,p_2)}(t^2) = x_{n-1}^{p_1+p_2} \left(\mu_{n-1} t^l\right)^{p_1-p_2} \cdot \mathrm{PE}\left[\sum_{i=1}^{l-1} \mu_{2i} t^{2i} + \mu_{n-1}^2 t^{2l}\right] , \tag{6.9a}$$

$$(2^n)^{II} : \qquad \mathrm{HWG}^{(2^n)}_{(p_1,p_2)}(t^2) = x_{n}^{p_1+p_2} \left(\mu_{n} t^l\right)^{p_1-p_2} \cdot \mathrm{PE}\left[\sum_{i=1}^{l-1} \mu_{2i} t^{2i} + \mu_{n}^2 t^{2l}\right] . \tag{6.9b}$$

Comparing this to the known results of [30], the symplectic resolution is of the form

$$\pi_{(2^n)} : T^*\left(\tfrac{\mathrm{SO}(2n)}{\mathrm{SU}(n)\times \mathrm{U}(1)}\right) \to \overline{\mathcal{O}}_{(2^n)} \tag{6.10}$$

The HWG indicates this resolution behaviour due to the prefactors μ_{n-1} or μ_n such that the corresponding HSS are the semi-simple orbits through μ_{n-1} or μ_n, respectively.

Remarks. Two remarks are in order: firstly, the accidental isomorphism $D_3 \cong A_3$ is manifest in the Hilbert series results. To see this, compare A-type (2^2) of table 3 with D-type $(3,1^3)$ of table 6, or compare A-type $(2,1^2)$ with D-type $(2^2,1^2)$. In both cases, the results agree upon identification of fugacity and weight labels.

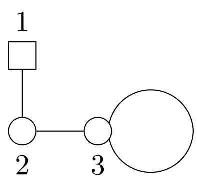
Secondly, for SO(8) there is a triality relating the Coulomb branch quivers $(3,1^5)$ and $(2^4)^{I/II}$ via outer automorphism on D_4, see for instance table 6. The triality rotates the fugacity and weight label of the node the flavour is attached to.

7 Exceptional algebras

Lastly, we consider the nilpotent orbits of the exceptional algebras, in particularly focusing on the characteristic height two examples of [4]. The Hilbert series and HWG for the singular Coulomb branch have been presented in [4] and, here, we compute the monopole formula in the presence of background charges. We summarise the results in tables 7 and 8.

***G_2* and *F_4*.** The quiver gauge theories of table 7 exhibit only a single U(1) flavour nodes such that there is no resolution parameter available on the Coulomb branch. This is consistent with the literature, see for instance [35, Proposition 3.21].

As a remark, there exists a Coulomb branch realisation



(7.1)

for the 10-dimensional nilpotent orbit $\mathcal{O}_{\{2,0\}}$ of G_2 although it has height larger than two. Although the U(1) flavour symmetry naively suggests that no symplectic resolution exists, it is known that the symplectic resolution is of the form $T^*(G_2/\mathrm{U}(2)) \to \overline{\mathcal{O}}_{\{2,0\}}$, see also [35]. This indicates that the Coulomb branch construction of nilpotent orbit closures $\overline{\mathcal{O}}$ with $\mathrm{ht}(\mathcal{O}) > 2$ is still an open issue for exceptional Lie algebras [4].

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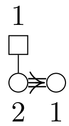
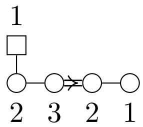
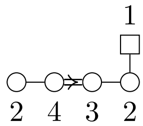
characteristic	$\dim_{\mathbb{C}}$	quiver	HWG with flux
$\{1,0\}$	6		no resolution
$\{1,0,0,0\}$	16		no resolution
$\{0,0,0,1\}$	22		no resolution

Table 7. Coulomb branch quiver gauge theories for the exceptional algebras G_2 and F_4: realisations of the (closures of the) nilpotent orbits of characteristic height 2.

E-type. Inspecting the E-series of table 8, we observe that only $\{1,0,0,0,1,0\}$ of E_6 and $\{0,0,0,0,0,2,0\}$ of E_7 admit non-trivial background charges on the Coulomb branch. All other Coulomb branches do not admit a resolution parameter, which is consistent with [35, Proposition 3.21].

Let us study $\{1,0,0,0,1,0\}$ of E_6 in more detail: again, we find two different behaviours of the HWG

$$\mathrm{HWG}^{\{1,0,0,0,1,0\}}_{(p_1,p_5)}(t^2) = \begin{cases} x_1^{p_1} x_5^{p_5} \left(\mu_5 t^2\right)^{p_1-p_5} \cdot \mathrm{PE}\left[\mu_6 t^2 + \mu_1\mu_5 t^4\right], & p_1 \geq p_5 \\ x_1^{p_1} x_5^{p_5} \left(\mu_1 t^2\right)^{p_5-p_1} \cdot \mathrm{PE}\left[\mu_6 t^2 + \mu_1\mu_5 t^4\right], & p_1 \leq p_5 \end{cases} \tag{7.2}$$

depending on the relative sign between the two U(1) flavour charges. As before, one can eliminate the overall unphysical U(1) in G_F via $p_1+p_5=0$ and reduce to a single (positive) resolution parameter $\propto \pm(p_1-p_5)$. The two different cases in (7.2) are a manifestation of the Mukai flop of E-type, see table 1. Moreover, the corresponding resolution is given by

$$\pi_{\{1,0,0,0,1,0\}} : T^*\left(\frac{E_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}\right) \to \overline{\mathcal{O}}_{\{1,0,0,0,1,0\}} \,. \tag{7.3}$$

We interpret the prefactor $\propto \mu_1, \mu_5$ as indicating the exceptional fibre $\pi^{-1}_{\{1,0,0,0,1,0\}}(0) \cong \frac{E_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}$, i.e. the resolution is given by the cotangent bundle of the HSS E_{III} of table 2.

Next, we consider $\{0,0,0,0,0,2,0\}$ of E_7: here, the Coulomb branch computation yields

$$\mathrm{HWG}^{\{0,0,0,0,0,2,0\}}_{(p_1,p_2)}(t^2) = x_6^{p_1+p_2}\left(\mu_6 t^3\right)^{p_1-p_2} \cdot \mathrm{PE}\left[\mu_1 t^2 + \mu_5 t^4 + \mu_6^2 t^6\right], \tag{7.4}$$

where one can eliminate the unphysical U(1) via $p_1+p_2=0$. Hence, one can reduce to a single p_1-p_2 background charge of SU(2), which gives the effective resolution parameter. The HWG (7.4) is consistent with the expected resolution

$$\pi_{\{0,0,0,0,0,2,0\}} : T^*\left(\frac{E_7}{E_6\times\mathrm{U}(1)}\right) \to \overline{\mathcal{O}}_{\{0,0,0,0,0,2,0\}} \,. \tag{7.5}$$

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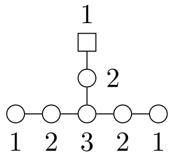
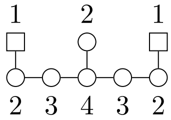
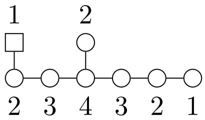
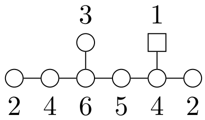
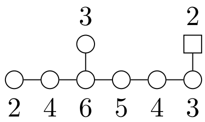
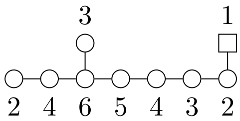
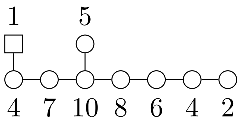
characteristic	$\dim_{\mathbb{C}}$	quiver	HWG with flux
$\{0,0,0,0,0,1\}$	22		no resolution
$\{1,0,0,0,1,0\}$	32		$\begin{cases} x_1^{p_1}x_5^{p_5}(\mu_5t^2)^{p_1-p_5}\mathrm{PE}[\mu_6t^2+\mu_1\mu_5t^4] & ,p_1\geq p_5 \\ x_1^{p_1}x_5^{p_5}(\mu_1t^2)^{p_5-p_1}\mathrm{PE}[\mu_6t^2+\mu_1\mu_5t^4] & ,p_1\leq p_5 \end{cases}$
$\{1,0,0,0,0,0,0\}$	34		no resolution
$\{0,0,0,0,1,0,0\}$	52		no resolution
$\{0,0,0,0,0,2,0\}$	54		$x_6^{p_1+p_2}(\mu_6t^3)^{p_1-p_2}\cdot\mathrm{PE}[\mu_1t^2+\mu_5t^4+\mu_6^2t^6]$
$\{0,0,0,0,0,0,1,0\}$	58		no resolution
$\{1,0,0,0,0,0,0,0\}$	92		no resolution

Table 8. Coulomb branch quiver gauge theories for the exceptional algebras E_6, E_7, E_8: realisations of the (closures of the) nilpotent orbits of characteristic height two. The unphysical U(1) in G_F can be eliminated as before: introducing an auxiliary fugacity z_0 for $\mathrm{U}(1)\subset G_F$, and imposing $z_0^{\sum_j p_j}\prod_{i=1}^n z_i^{r_i}=1$, with r_i the ranks of the gauge nodes, leads to $x_1x_5=1$ for $\{1,0,0,0,1,0\}$ and $x_6=1$ for $\{0,0,0,0,0,2,0\}$ (and $z_0\equiv 1$).

We note in particular that there exists only one resolution and that the prefactor $\propto \mu_6$ suggests to be interpreted as manifestation of the exceptional fibre $\pi^{-1}_{\{0,0,0,0,0,2,0\}}(0)\cong\frac{E_7}{E_6\times\mathrm{U}(1)}$. This follows from the observation that the sixth fundamental weight of E_7 has stabiliser $E_6\times\mathrm{U}(1)$ in E_7.

Therefore, all nilpotent orbits of characteristic height two that allow for a symplectic resolution are resolved by cotangent bundles of the Hermitian symmetric spaces E_{III} and E_{VII}. The Coulomb branch construction via unitary quivers and its resolution via the monopole formula with background charges reproduce these features consistently.

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Remark. So far, the resolutions of height two orbit closures have exhausted all Hermitian symmetric spaces, but has not produced all possible basic Mukai flops of table 1. The missing piece would be the 50-dimensional E_6 orbit of characteristic $\{0,1,0,1,0,0\}$, but it is not of characteristic height two such that no unitary quiver realisation is known [4].

8 Conclusions

In this paper we have examined to what extent the prescription of the monopole formula with background charges is suitable to study the resolutions of certain Coulomb branches, which are nilpotent orbit closures of (characteristic) height two. For the examples considered with $T^*(G/P) \to \overline{\mathcal{O}}$ such that $G/P \cong \mathcal{O}^{\text{ss}}_{\mu}$, the HWG takes a remarkably simple form

$$\text{HWG}_{\text{flux}} = \left(\mu\, t^{\Delta}\right)^{\text{flux}} \cdot \text{HWG}_{\text{singular}}\,, \tag{8.1}$$

which allows us to show that the monopole formula is consistent with the following known facts about resolutions of $\overline{\mathcal{O}}$:

(i) Number of resolution parameters: if the flavour symmetry is a single U(1) then the flux associated can simply be absorbed by a redefinition of the GNO magnetic weights. Hence, the monopole formula does not admit any resolution parameter. If the flavour symmetry group is larger, there exists a non-trivial resolution. We note in particular, that an overall U(1) in G_F is unphysical; for instance, the resolution parameter for G_F=(U(1)×U(1))/U(1) is either $p_1 - p_2 \geq 0$ or $p_2 - p_1 \geq 0$, while for G_F=U(2)/U(1) the resolution parameter is $p_1 - p_2 \geq 0$.

(ii) Form of resolution: the monopole formula results indicate that the exceptional fibre of the resolution can be read off from the prefactor in (8.1). In more detail, if $\overline{\mathcal{O}}$ is resolved by the cotangent bundle $T^*(G/P)$ of a HSS G/P, then the prefactor is proportional to a single fundamental weight μ, such that the HSS is realised as semi-simple orbit $\mathcal{O}^{\text{ss}}_{\mu}$ through μ in Lie(G). Moreover, the size of the coset G/P, as base manifold of the resolved space, is determined by the exponent of $t^{\Delta\cdot\text{flux}}$.

(iii) Existence of multiple resolutions, i.e. Mukai flops: the cases with two U(1) flavour nodes allow for two distinct cases, depending on which U(1) flux is larger. The HWG becomes case dependent and we interpret this as the manifestation of the basic Mukai flops. The examples considered reproduce all but one of the possible basic flops, see table 1. The exception $E_{6,II}$ does not correspond to a height two nilpotent orbit.

As a remark, existence of symplectic resolutions and, in case of multiple resolutions, their relation via (sequences of) Mukai flops can equivalently be discussed via weighted Dynkin diagrams. A brief summary is provided in appendix B.

Additionally, it is interesting to note that all Hermitian symmetric spaces of table 2 are realised as base spaces for the resolutions of the (characteristic) height two nilpotent orbits. We summarise the orbits considered and their resolutions in tables 9–10.

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type	partition	resolution
A_n	$\rho=(2^k,1^{n+1-2k})$, $2\leq 2k\leq n+1$	$T^*\left(\frac{\mathrm{SU}(n+1)}{\mathrm{S}(\mathrm{U}(n+1-k)\times\mathrm{U}(k))}\right)\to\overline{\mathcal{O}}_\rho$
B_n	$\rho=(2^{2k},1^{2n+1-4k})$, $k>0$	—
	$\rho=(3,1^{2n-2})$	$T^*\left(\frac{\mathrm{SO}(2n+1)}{\mathrm{SO}(2n-1)\times\mathrm{SO}(2)}\right)\to\overline{\mathcal{O}}_\rho$
C_n	$\rho=(2^k,1^{2(n-k)})$, $n>k\geq 1$	—
	$\rho=(2^n)$	$T^*\left(\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}\right)\to\overline{\mathcal{O}}_\rho$
D_n	$\rho=(2^{2k},1^{2n-4k})$, $n-1>2k\geq 2$	—
	$\rho=(2^{n-1},1^2)$, $n=\text{odd}$ $\rho=(2^n)$, $n=\text{even}$	$\Big\}$ $T^*\left(\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}\right)\to\overline{\mathcal{O}}_\rho$
	$\rho=(3,1^{2n-3})$	$T^*\left(\frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)\times\mathrm{SO}(2)}\right)\to\overline{\mathcal{O}}_\rho$

Table 9. Summary of resolutions of height two nilpotent orbits for classical algebras.

type	characteristic	resolution
G_2	$\{1,0\}$	—
F_4	$\{1,0,0,0\}$	—
	$\{0,0,0,1\}$	—
E_6	$\{0,0,0,0,0,1\}$	—
	$\{1,0,0,0,1,0\}$	$T^*\left(\frac{E_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}\right)\to\overline{\mathcal{O}}_{\{1,0,0,0,1,0\}}$
E_7	$\{1,0,0,0,0,0,0\}$	—
	$\{0,0,0,0,1,0,0\}$	—
	$\{0,0,0,0,0,2,0\}$	$T^*\left(\frac{E_7}{E_6\times\mathrm{U}(1)}\right)\to\overline{\mathcal{O}}_{\{0,0,0,0,0,2,0\}}$
E_8	$\{0,0,0,0,0,0,1,0\}$	—
	$\{1,0,0,0,0,0,0,0\}$	—

Table 10. Summary of resolutions of characteristic height two nilpotent orbits for exceptional algebras.

In summary, the monopole formula, in combination with the unitary Coulomb branch quiver realisations for the BCD-type and exceptional nilpotent orbit closures of (characteristic) height two, exhibits all features of the symplectic resolutions correctly. Thus, the monopole formula with background charges is not only suitable for gluing techniques as indicated in [17], but is in fact a tool to study the geometry of the resolved spaces. On the other hand, the inclusion of background fluxes opens a window to study the 3-dimensional $\mathcal{N}=4$ theories with discrete real masses.

Outlook. Coulomb branches of nilpotent orbit closures with $\mathrm{ht}(\mathcal{O})\geq 3$ have not been considered here. Let us mention some of the generalisations which are to be expected. The flavour symmetry groups may become larger such that more than one resolution parameter

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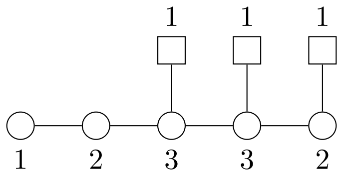
becomes relevant. In this case, the different orderings of the fluxes will all be related by a locally trivial family of Mukai flops. To put it differently, the basic Mukai flops will not be sufficient to relate the different symplectic resolutions of a given orbit, but a finite family of Mukai flops will do.

As example, consider $\overline{\mathcal{O}}_{(3,2,1)}$ of A_5 with Coulomb branch quiver



(8.2)

where the three U(1) fluxes p_4, p_5, p_6 have six ordered permutations $p_{\sigma(4)} \geq p_{\sigma(5)} \geq p_{\sigma(6)}$. From this we expect six different resolutions of $\overline{\mathcal{O}}_{(3,2,1)}$. In fact, comparing this to [25, Example 4.6] there exist six polarisations $P_{\sigma(4),\sigma(5),\sigma(6)}$. Each gives rise to a resolution $T^*(\mathrm{SU}(6)/P_{\sigma(4),\sigma(5),\sigma(6)}) \to \overline{\mathcal{O}}_{(3,2,1)}$, and any two are linked by birational maps in the form of a locally trivial family of Mukai flops of type A. However, we do not expect the HWG with fluxes to respect the appealing form (8.1), simply because the HWG without fluxes does not have a simple PE anymore.

In addition, in [4, section 3.1] a Hilbert series formula for normalisations of nilpotent orbit closures has been proposed. In view of our final result (8.1), the formula seems suited to deal with resolutions which do not come from a simple flag variety. This localisation approach might also be able to compute the E_6 orbit of characteristic $\{0,1,0,1,0,0\}$, which is expected to realise the last basic Mukai flop $E_{6,II}$.

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A Conventions

In this appendix, we provide the conventions used in all calculations. We note that all Coulomb branch quivers T_Γ are balanced and the set of balanced nodes forms exactly the

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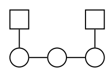
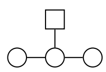
Dynkin diagram Γ of the algebra $\mathfrak{g}$ to which the nilpotent orbit closure $\mathcal{M}_C(T_\Gamma) \cong \overline{\mathcal{O}}$ belongs. As such, the quiver T_Γ has exactly $r = \mathrm{rk}(G)$ many unitary nodes, each node $i = 1, \ldots, r$ is associated with a topological symmetry group $\mathrm{U}(1)_i$, which we count by a fugacity z_i. The root space fugacities z_i are transformed into the weight space fugacities x_i by means of the Cartan matrix A_{ij} of the algebra $\mathfrak{g}$; in detail,

$$z_i = \prod_{j=1}^{r} x_j^{A_{ij}} \qquad \Leftrightarrow \qquad x_i = \prod_{j=1}^{r} z_j^{(A^{-1})_{ij}} \,. \tag{A.1}$$

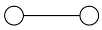
Labelling of fugacities z_i (or likewise x_i) and background charges p will follow the numbering of the nodes in the corresponding Dynkin diagram, see for instance [60, table IV]. As example

U(1)-flavours: 

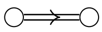
or U(2)-flavour: 

. (A.2)

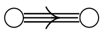
In the relevant quivers, the gauge nodes are linked by different types of hyper multiplets, which have the following contributions to the conformal dimension:



$$\leftrightarrow \qquad \Delta_{\text{h-plet}} = \frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{l} |q_{1,n} - q_{2,m}| \,, \tag{A.3a}$$



$$\leftrightarrow \qquad \Delta_{\text{h-plet}} = \frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{l} |2 \cdot q_{1,n} - q_{2,m}| \,, \tag{A.3b}$$



$$\leftrightarrow \qquad \Delta_{\text{h-plet}} = \frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{l} |3 \cdot q_{1,n} - q_{2,m}| \,. \tag{A.3c}$$

Here, the magnetic charges of the i-th node $\mathrm{U}(k_i)$ are labelled as $q_{i,n}$ for $n = 1, \ldots, k_i$. The non-simply laced links of (A.3b)–(A.3c) have been introduced in [48, equation 3.3].

When working with Hilbert series (HS) and Highest Weight Generating functions (HWG), important tools are the Plethystic Exponential (PE) and Plethystic Logarithm (PL), the inverse of the PE. For our purpose, it is sufficient to note

$$\mathrm{PE}\left[\sum_{i=1}^{c} f_i t^{a_i} - \sum_{j=1}^{d} g_j t^{b_j}\right] \equiv \frac{\prod_{j=1}^{d}(1 - g_j t^{b_j})}{\prod_{i=1}^{c}(1 - f_i t^{a_i})} \,, \tag{A.4}$$

where $a_i, b_j \in \mathbb{N}$ are exponents and f_j, g_j are monomials in weight or root fugacities. For further details, we refer to [11].

B Weighted Dynkin diagram

Another method of labelling nilpotent orbits of $\mathfrak{g}$ employs weighted Dynkin diagrams (WDD), which decorate the nodes of the Dynkin diagram for $\mathfrak{g}$ with labels $0, 1$, or 2. We

partition	WDD	MDD
(4)	○–○–○ 2 2 2	●–●–●
(3, 1)	○–○–○ 2 0 2	○–●–● or ●–○–● or ●–●–○
(2^2)	○–○–○ 0 2 0	○–●–○
$(2, 1^2)$	○–○–○ 1 0 1	●–○–○ or ○–○–●
(1^4)	○–○–○ 0 0 0	○–○–○

Table 11. Nilpotent orbits of A_3 can be labelled by partitions or weighted Dynkin diagrams. The corresponding parabolic subalgebras can be encoded in marked Dynkin diagrams.

refer to [20] for details. In contrast to partitions, WDDs are applicable to nilpotent orbits of classical as well as exceptional Lie algebras. On the other hand, the set of $ABCD$-type partitions is one-to-one to all $ABCD$-type nilpotent orbits; whereas, the number $3^{\mathrm{rk}(G)}$ of possible WDDs is significantly larger than the set of nilpotent orbits. To our understanding, no algorithm exists that could predict which diagrams are realised.

To provide a more intuitive picture, all unitary Coulomb branch realisations of height two nilpotent orbit closures are built from the corresponding WDD in the following way: the WDD provides the ranks of the flavour nodes and the condition for the quiver to be balanced determines the ranks of the gauge nodes uniquely.

Following [37], one introduces the sets Θ_i, which collect all nodes in the WDD with label $i = 0, 1, 2$. Focusing on height two, a result by [37, Lemma 3.2] states the following: ABC-type nilpotent orbit closures admit a symplectic resolution if and only if the number of elements in Θ_1 is even. For D-type orbit closures, a symplectic resolution exists if and only if either the number of elements in Θ_1 is even, or the number of elements in Θ_1 is even and the two spinor nodes belong to Θ_1.

The weighted Dynkin diagram allows to deduce the *standard parabolic subalgebra* associated to the nilpotent orbit, and the set of all parabolic subalgebras can be labelled by marked Dynkin diagrams. As $|\Theta_1|$ is necessarily even for orbit closures that can be resolved, one can subdivide the set into two sets of equal length and try to construct two different parabolic subalgebras. This leads to [37, Theorem 3.3]: (i) two symplectic resolutions for A-type, (ii) one symplectic resolution for BC-type, and (iii) two symplectic resolutions for D-type if the two spinor nodes are contained in Θ_1, otherwise there exists only one resolution.

As an example, the nilpotent orbits of A_3, labelled by their weighted Dynkin diagrams, and the corresponding parabolic subalgebras, labelled by marked Dynkin diagrams (MDD), are summarised in table 11. All (non-trivial) orbits of A_3 admit symplectic resolutions, as there are either zero nodes with weight label 1 or exactly two. The existence of multiple resolutions for a single orbit closure can be observed from the marked Dynkin diagrams. A way to relate all possible parabolic subalgebras that give rise to different resolutions of the

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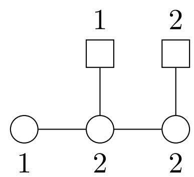
same orbit closure is presented in [61, Definition 1] by means of operations on the MDD. Hence, the two (dual) MDDs of $(2,1^2)$ are the basic A-type Mukai flop, while the three MDDs of $(3,1)$ can be obtain by locally applying the A-type Mukai flop of A_2, i.e. the dual MDDs are ●–○ and ○–● . Therefore, $\overline{\mathcal{O}}_{(3,1)}$ of A_3 admits three different resolutions, which can also be seen from the corresponding Coulomb branch quiver



(B.1)

in which one would identify the three resolutions by the relative size of the U(1) flux p_1 with respect to the U(2) fluxes $p_2 \geq p_3$.

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