



# Sharp Results on Sampling with Derivatives in Shift-Invariant Spaces and Multi-Window Gabor Frames

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## Abstract

We study the problem of sampling with derivatives in shift-invariant spaces generated by totally-positive functions of Gaussian type or by the hyperbolic secant. We provide sharp conditions in terms of weighted Beurling densities. As a by-product we derive new results about multi-window Gabor frames with respect to vectors of Hermite functions or totally positive functions.

**Keywords** Shift-invariant space · Sampling with derivatives · Gaussian · Totally positive function · Jensen's formula · Gabor frame · Beurling density

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## 1 Introduction and Results

In the problem of sampling with derivatives, one tries to recover or approximate a function by sampling a number of its derivatives. Analogously to Hermite interpolation,

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this procedure is sometimes called Hermite sampling. For a well-defined problem, one must fix a suitable signal model, which in engineering is usually a space of bandlimited functions (the Paley–Wiener space in mathematical terminology). In recent years the more general model of shift-invariant spaces has received considerable attention as a viable substitute for bandlimited functions. See [4] for an early survey.

Hermite sampling can be seen as a purely mathematical problem in approximation theory, but it is also informed by practical considerations. Whereas a sample  $f(\lambda)$  at a sampling point  $\lambda$  gives its pointwise value, the derivative  $f'(\lambda)$  measures the trend of  $f$  at  $\lambda$ , and higher derivatives yield information about the local approximation by Taylor polynomials. In addition, by taking several measurements at each point, one may hope to use fewer sampling points.

Based on the experience gained in [15], we will analyze Hermite sampling in shift-invariant spaces that are generated by certain totally positive functions. We will call a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  *totally positive of Gaussian type* if its Fourier transform factors as

$$\hat{g}(\xi) = \prod_{j=1}^n (1 + 2\pi i \delta_j \xi)^{-1} e^{-c\xi^2}, \quad \delta_1, \dots, \delta_n \in \mathbb{R}, c > 0, n \in \mathbb{N} \cup \{0\}. \quad (1.1)$$

We study the problem of sampling with multiplicities in the shift-invariant space

$$V^p(g) = \left\{ f \in L^p(\mathbb{R}) : f = \sum_{k \in \mathbb{Z}} c_k g(\cdot - k), c \in \ell^p(\mathbb{Z}) \right\},$$

generated by a totally-positive function of Gaussian-type, where  $1 \leq p \leq \infty$ . To describe the sampling process, we fix a sampling set  $\Lambda \subseteq \mathbb{R}$  and a multiplicity function  $m_\Lambda : \Lambda \rightarrow \mathbb{N}$ , and call  $(\Lambda, m_\Lambda)$  a set with multiplicity. The number  $m_\Lambda(\lambda)$  indicates how many derivatives are sampled at  $\lambda \in \Lambda$ .

We then say that  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^p(g)$  with  $1 \leq p < \infty$  if there exist constants  $A, B > 0$  such that

$$A \|f\|_p^p \leq \sum_{\lambda \in \Lambda} \sum_{j=0}^{m_\Lambda(\lambda)-1} |f^{(j)}(\lambda)|^p \leq B \|f\|_p^p, \quad f \in V^p(g). \quad (1.2)$$

If  $p = \infty$ , a sampling set is defined by the inequalities

$$A \|f\|_\infty \leq \sup_{\lambda \in \Lambda} \max_{0 \leq j \leq m_\Lambda(\lambda)-1} |f^{(j)}(\lambda)| \leq B \|f\|_\infty, \quad f \in V^\infty(g). \quad (1.3)$$

From a theoretical point of view, the sampling inequality (1.2) completely solves the (Hermite) sampling problem. We note that a sampling inequality always leads to a general reconstruction algorithm based on frame theory [11]. In addition, for localized generators, the frame algorithm converges even in the correct  $L^p$ -norm [10]. Thus (1.2) is also a first step towards the numerical treatment of the sampling problem.

Our objective is the characterization of sampling sets satisfying the sampling inequality (1.2) and to obtain sharp conditions on the sampling set. In Beurling's tra-

dition of complex analysis, we will characterize sampling sets in terms of a weighted version of Beurling's lower density

$$D^-(\Lambda, m_\Lambda) := \liminf_{r \rightarrow \infty} \inf_{x \in \mathbb{R}} \frac{1}{2r} \sum_{\lambda \in \Lambda \cap [x-r, x+r]} m_\Lambda(\lambda). \quad (1.4)$$

Within this setting we can already formulate our main result.

**Theorem 1.1** *Let  $g$  be a totally positive function of Gaussian type. Let  $\Lambda \subseteq \mathbb{R}$  be a separated set, and let  $m_\Lambda : \Lambda \rightarrow \mathbb{N}$  be a multiplicity function such that  $\sup_{\lambda \in \Lambda} m_\Lambda(\lambda) < \infty$ .*

- (i) *If  $D^-(\Lambda, m_\Lambda) > 1$ , then  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^p(g)$  and every  $1 \leq p \leq \infty$ .*
- (ii) *Conversely, if  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^2(g)$ , then  $D^-(\Lambda, m_\Lambda) \geq 1$ .*

Theorem 1.1 extends one of the results in [15] to sampling with multiplicities. We also have an analogous density result for the shift-invariant space generated by the hyperbolic secant.

**Theorem 1.2** *Let  $\psi(x) = \operatorname{sech}(ax) = \frac{2}{e^{ax} + e^{-ax}}$  be the hyperbolic secant. Let  $\Lambda \subseteq \mathbb{R}$  be a separated set and  $m_\Lambda$  be a multiplicity function such that  $\sup_{\lambda \in \Lambda} m_\Lambda(\lambda) < \infty$ .*

- (i) *If  $D^-(\Lambda, m_\Lambda) > 1$ , then  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^p(\psi)$  and every  $1 \leq p \leq \infty$ .*
- (ii) *Conversely, if  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^2(\psi)$ , then  $D^-(\Lambda, m_\Lambda) \geq 1$ .*

For comparison, we state the corresponding sampling result for the Paley–Wiener space

$$\operatorname{PW}^2 = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]\}.$$

The statement is analogous to Theorems 1.1 and 1.2 and is considered folklore among complex analysts (we tested it!).

**Theorem 1.3** *Let  $\Lambda \subseteq \mathbb{R}$  be a separated set, and let  $m_\Lambda$  be a multiplicity function such that  $\sup_{\lambda \in \Lambda} m_\Lambda(\lambda) < \infty$ .*

- (i) *If  $D^-(\Lambda, m_\Lambda) > 1$ , then  $(\Lambda, m_\Lambda)$  is a sampling set for  $\operatorname{PW}^2$ .*
- (ii) *Conversely, if  $(\Lambda, m_\Lambda)$  is a sampling set for  $\operatorname{PW}^2$ , then  $D^-(\Lambda, m_\Lambda) \geq 1$ .*

Although folklore, Theorem 1.3 does not seem to have been formulated explicitly in the literature. A very interesting result involving divided differences of samples was proved for the Bernstein space  $\operatorname{PW}^\infty$  by Lyubarski and Ortega-Cerda [18]. For the Fock space, a result similar to Theorem 1.3 was derived early on by Brekke and Seip [9].

Theorems 1.1 and 1.2 have also several consequences for Gabor systems. Specifically, we characterize semi-regular sets  $\Lambda \times \beta\mathbb{Z}$  that generate a multiwindow Gabor

frame with respect to the first  $n$  Hermite functions or with respect to a specific finite set of totally positive functions. See Sect. 6 for the precise formulations.

In the literature, most sampling results for shift-invariant spaces work with the assumption that the sampling set  $\Lambda$  is “dense enough”. However, when the sufficient density is made explicit, it is usually very far from the known necessary density, even in dimension 1. In fact, until [15], all authors use the covering density or maximum gap between samples, and the density then depends on some modulus of continuity of the generator. See [3] for one of the first nonuniform sampling theorems in shift-invariant spaces, [21] for nonuniform sampling with derivatives for bandlimited functions, and [2,5,23] for more recent examples of sufficient conditions for Hermite sampling in terms of the covering density.

In light of [15], the sharp results for sampling with derivatives are perhaps not surprising, but they definitely go far beyond the current state of the art. Our main point is to show the usefulness and power of the established methods, which consist of the combination of Beurling’s techniques, spectral invariance, complex analysis, and the comparison of zero sets in different shift-invariant spaces. We believe that these methods carry a high potential in many other situations.

To arrive at sharp results, we combine several techniques. Roughly, we proceed in three steps:

- (i) We use Beurling’s method of weak limits and show that the sampling inequality (1.3) for  $p = \infty$  is equivalent to the fact that every weak limit of integer translates of  $\Lambda$  is a uniqueness set for  $V^\infty(g)$ . In this way, we obtain a general characterization of sampling sets *without inequalities* (Theorem 3.4).
- (ii) To switch between sampling inequalities for  $p = \infty$  and  $p < \infty$ , we use the theory of localized frames and Sjöstrand’s beautiful version of Wiener’s lemma for convolution-dominated matrices [25]. These two steps are part of a general mathematical formalism that can be applied to many different situations. In particular, they work for shift-invariant spaces with almost arbitrary generators.
- (iii) The concrete understanding then rests on the analysis of uniqueness sets for a particular shift-invariant space  $V^p(g)$ , or in other words, we need to analyze the zero sets of arbitrary functions in  $V^p(g)$ . For instance, for the classical Paley–Wiener space, this is the relation between the density of the zero set of an entire function and its growth. This is precisely the aspect where we develop new arguments. Firstly, we observe that every function in  $V^p(\phi)$  for a Gaussian generator  $\phi$  possesses an extension to an entire function, and secondly, we can relate the real zeros of some  $f \in V^p(\phi)$  to the complex zeros of its analytic extension. A similar but technically more involved strategy works for the hyperbolic secant  $\psi(x) = 2(e^{ax} + e^{-ax})^{-1}$ . In a final step, we relate the zero sets of functions in *different* shift-invariant spaces to each other. In this way we develop a direct line of arguments and avoid the detour in [15] via the characterization of Gaussian Gabor frames.

The paper is organized as follows: Sect. 2 introduces the necessary definitions for sampling in vector-valued shift-invariant spaces. These provide a convenient language to formulate the problem of sampling with derivatives. Section 3 then contains the main structural characterization of sampling with derivatives and the necessary density

condition (Proposition 3.7). Section 4 is devoted to the investigation of the density of zero sets in shift-invariant spaces. This is the part that contains the main arguments and new proof ideas. The proofs of Theorems 1.1, 1.2, and 1.3 are then in Sect. 5. In Sect. 6 we draw some consequences of the sampling theorems with derivatives for multi-window Gabor frames. Finally Sect. 7 contains some of the postponed proofs of the structural results in Sects. 2 and 3. As these are essentially known, we explain only the necessary modifications.

## 2 Vector-Valued Shift-Invariant Spaces and Sampling

### 2.1 Vector-Valued Shift-Invariant Spaces

The treatment of sampling with derivatives requires us to formulate several standard concepts for vector-valued functions. In this section, we collect the precise definitions. For the proper formulation of sampling results, we make use of the Wiener amalgam space  $W_0 = W_0(\mathbb{R})$ , which consists of continuous functions  $g$  such that

$$\|g\|_W := \sum_{k \in \mathbb{Z}} \max_{x \in [k, k+1]} |g(x)| < \infty.$$

Let  $G = (G^1, \dots, G^N) \in (W_0(\mathbb{R}))^N$ . We consider the vector-valued shift-invariant space

$$V^p(G) := \left\{ \sum_{k \in \mathbb{Z}} c_k G(\cdot - k) : c \in \ell^p(\mathbb{Z}) \right\} \quad (2.1)$$

as a subspace of  $(L^p(\mathbb{R}))^N$  with norm

$$\|(F^1, \dots, F^N)\|_p := \left( \sum_{j=1}^N \|F^j\|_p^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and  $\|(F^1, \dots, F^N)\|_\infty = \max_{j=1, \dots, N} \|F^j\|_\infty$ . We always assume that  $G$  has stable integer shifts, i.e.,

$$\left\| \sum_{k \in \mathbb{Z}} c_k G(\cdot - k) \right\|_p \asymp \|c\|_p. \quad (2.2)$$

### 2.2 Sampling and Weak Limits

We consider tuples of sets  $\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^N)$  with  $\Lambda^j \subseteq \mathbb{R}$ . We say that  $\vec{\Lambda}$  is a *sampling set* for  $V^p(G)$ ,  $1 \leq p \leq \infty$ , if

$$\|F\|_p \asymp \left( \sum_{j=1}^N \|F^j|_{\Lambda^j}\|_p^p \right)^{1/p} = \left( \sum_{j=1}^N \sum_{\lambda \in \Lambda^j} |F^j(\lambda)|^p \right)^{1/p}, \quad \text{for all } F \in V^p(G).$$

For  $p = \infty$ , the condition reads as  $\|F\|_\infty \asymp \max_{j=1,\dots,N} \|F^j|_{\Lambda^j}\|_\infty$ . We say that  $\vec{\Lambda}$  is a *uniqueness set* for  $V^p(G)$  if whenever  $F \in V^p(G)$  is such that  $F^j \equiv 0$  on  $\Lambda^j$ , for all  $j = 1, \dots, N$ , then  $F \equiv 0$ . Clearly, sampling sets are also uniqueness sets.

We first recall Beurling's notion of a weak limit of a sequence of sets. A sequence  $\{\Lambda_n : n \geq 1\}$  of subsets of  $\mathbb{R}$  is said to *converge weakly* to a set  $\Lambda \subseteq \mathbb{R}$ , denoted by  $\Lambda_n \xrightarrow{w} \Lambda$ , if for every open bounded interval  $(a, b)$  and every  $\varepsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,

$$\Lambda_n \cap (a, b) \subseteq \Lambda + (-\varepsilon, \varepsilon) \text{ and } \Lambda \cap (a, b) \subseteq \Lambda_n + (-\varepsilon, \varepsilon).$$

We let  $W_{\mathbb{Z}}(\Lambda)$  denote the class of all sets  $\Gamma$  that can be obtained as weak limits of integer translates of  $\Lambda$ , i.e.,  $\Gamma \in W_{\mathbb{Z}}(\Lambda)$  if there exists a sequence  $\{k_n : n \geq 1\} \subseteq \mathbb{Z}$  such that  $\Lambda + k_n \xrightarrow{w} \Gamma$ . We extend this notion to tuples of sets as follows. Given two  $N$ -tuples of sets  $\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^N)$  and  $\vec{\Gamma} = (\Gamma^1, \dots, \Gamma^N)$ , we say that  $\vec{\Gamma} \in W_{\mathbb{Z}}(\vec{\Lambda})$  if there exists a sequence  $\{k_n : n \geq 1\} \subseteq \mathbb{Z}$  such that  $\Lambda^j + k_n \xrightarrow{w} \Gamma^j$  for all  $1 \leq j \leq N$ . (Note that the limits involve *the same sequence*  $\{k_n : n \geq 1\}$  for all  $j$ .)

The following is a vector-valued extension of [15, Theorem 3.1].

**Theorem 2.1** *Let  $G = (G^1, \dots, G^N) \in (W_0(\mathbb{R}))^N$  have stable integer shifts, and let  $\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^N)$  be a tuple of separated sets. Then the following are equivalent.*

- (a)  $\vec{\Lambda}$  is a sampling set for  $V^p(G)$  for some  $p \in [1, \infty]$ .
- (b)  $\vec{\Lambda}$  is a sampling set for  $V^p(G)$  for all  $p \in [1, \infty]$ .
- (c) Every weak limit  $\vec{\Gamma} \in W_{\mathbb{Z}}(\vec{\Lambda})$  is a sampling set for  $V^\infty(G)$ .
- (d) Every weak limit  $\vec{\Gamma} \in W_{\mathbb{Z}}(\vec{\Lambda})$  is a set of uniqueness for  $V^\infty(G)$ .

The proof is similar to the scalar-valued version; a sketch of the proof is given in Sect. 7.

### 3 Sampling with Multiplicities

#### 3.1 Sets with Multiplicities and Derivatives

For  $N \in \mathbb{N}$ , we let  $W_0^N = W_0^N(\mathbb{R})$  be the class of functions  $g$  having derivatives up to order  $N - 1$  in  $W_0(\mathbb{R})$ . For a set with multiplicity  $(\Lambda, m_\Lambda)$ , we define its height as  $\sup_\lambda m_\Lambda(\lambda)$ . When sampling in shift-invariant spaces with generators on  $W_0^N(\mathbb{R})$  we assume that the sampling sets have height  $\leq N$ . The lower density of  $(\Lambda, m_\Lambda)$  is defined by (1.4).

#### 3.2 Sampling with Derivatives

We now describe how the problem of sampling with multiplicities can be reformulated in terms of sampling of vector-valued functions.

Let a generator  $g \in W_0^N(\mathbb{R})$  with stable integer shifts be given. We define  $G \in (W_0(\mathbb{R}))^N$  by choosing as components the derivatives of  $g$ , so

$$G = (g, g^{(1)}, \dots, g^{(N-1)}).$$

There is an obvious one-to-one correspondence between  $f = \sum_k c_k g(\cdot - k) \in V^p(g)$  and  $F = (f, f^{(1)}, \dots, f^{(N-1)}) \in V^p(G)$ . In addition, since  $g$  has stable integer shifts, we have the norm equivalence

$$\|f\|_p \asymp \|c\|_p.$$

Furthermore, since  $g^{(j)} \in W_0(\mathbb{R})$  for  $1 \leq j \leq N-1$ , there is a constant  $B > 0$  such that

$$\|f^{(j)}\|_p \leq B\|c\|_p \quad \text{for } 1 \leq j \leq N-1,$$

and this implies

$$\|f\|_p \asymp \|c\|_p \asymp \|F\|_p.$$

This shows that  $G$  has stable integer shifts in the sense of (2.2).

Second, given a set with multiplicity  $(\Lambda, m_\Lambda)$  and height at most  $N < \infty$ , we consider the tuple sets  $\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^N)$  given by

$$\Lambda^k := \{\lambda \in \Lambda : m_\Lambda(\lambda) \geq k\}.$$

Note that  $\Lambda^1 = \Lambda$ . The connection between vector-valued sampling and sampling with derivatives is stated in the following lemma, which is a direct consequence of our notation.

**Lemma 3.1** *A set with multiplicity  $(\Lambda, m_\Lambda)$  and height at most  $N < \infty$  is a sampling set for  $V^p(g)$  in the sense of (1.2) if and only if  $\vec{\Lambda} = (\Lambda^1, \dots, \Lambda^N)$  is a sampling set for  $V^p(G)$ , with  $G = (g, g^{(1)}, \dots, g^{(N-1)})$ .*

Finally, we interpret a weak limit  $\vec{\Gamma} \in W_{\mathbb{Z}}(\vec{\Lambda})$  as a set with multiplicity by setting  $\Gamma := \Gamma^1$  and

$$m_\Gamma(\gamma) := \max\{j \in \mathbb{N} : \gamma \in \Gamma^j\}, \quad \gamma \in \Gamma.$$

In order to keep our notation consistent, we also write  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$  for the current situation.

For *separated sets*  $\Lambda$ , i.e.,  $\inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'\} > 0$ , we have the following alternative description of weak convergence.

**Proposition 3.2** *Let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and finite height  $N$ , let  $(\Gamma, m_\Gamma)$  be a set with multiplicity, and  $\{k_n : n \geq 1\} \subseteq \mathbb{Z}$ . Then  $\Lambda^j - k_n \xrightarrow{w} \Gamma^j$ , as  $n \rightarrow \infty$  for all  $j = 1, \dots, N$  if and only if*

$$\sum_{\lambda \in \Lambda} m_\Lambda(\lambda) \delta_{\lambda - k_n} \rightarrow \sum_{\gamma \in \Gamma} m_\Gamma(\gamma) \delta_\gamma, \quad \text{as } n \rightarrow \infty,$$

in the  $\sigma(C_c^*, C_c)$  topology (where  $C_c$  denotes the class of continuous functions with compact support).

A proof of Proposition 3.2 is given in Sect. 7. As a consequence, we obtain the following lemma; see, e.g., [15, Lemma 7.1] for a proof without multiplicities.

**Lemma 3.3** *Let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and finite height, and let  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$ . Then  $D^-(\Gamma, m_\Gamma) \geq D^-(\Lambda, m_\Lambda)$ .*

### 3.3 Characterization of Sampling with Derivatives

Theorem 2.1 can be recast in terms of sampling with derivatives.

**Theorem 3.4** *Let  $g \in W_0^N(\mathbb{R})$  have stable integer shifts, and let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and height at most  $N < \infty$ . Then the following are equivalent.*

- (a)  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^p(g)$  for some  $p \in [1, \infty]$ .
- (b)  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^p(g)$  for all  $p \in [1, \infty]$ .
- (c) Every weak limit  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$  is a sampling set for  $V^\infty(g)$ .
- (d) Every weak limit  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$  is a set of uniqueness for  $V^\infty(g)$ .

For bandlimited functions, only some of the implications in Theorem 3.4 are valid. These are formulated in terms of the Bernstein space  $\text{PW}^\infty$  of continuous bounded functions that are Fourier transforms of distributions supported on  $[-1/2, 1/2]$ .

**Theorem 3.5** *Let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and finite height. Then the following are equivalent.*

- (a)  $(\Lambda, m_\Lambda)$  is a sampling set for  $\text{PW}^\infty$ .
- (c) Every weak limit  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$  is a sampling set for  $\text{PW}^\infty$ .
- (d) Every weak limit  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$  is a set of uniqueness for  $\text{PW}^\infty$ .

As a replacement for the  $L^2$  part of Theorem 3.4, we have the following result.

**Proposition 3.6** *Let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and finite height, and assume that  $(\Lambda, m_\Lambda)$  is a sampling set for  $\text{PW}^\infty$ . Then, for every  $\alpha \in (0, 1)$ ,  $(\alpha\Lambda, m_\Lambda)$  is a sampling set for  $\text{PW}^2$ .*

Theorem 3.5 and Proposition 3.6 are due to Beurling [7,8] (without multiplicities) - see also [19, Theorem 2.1]. A slight modification of the arguments yields the case with multiplicities.

### 3.4 Necessary Density Conditions

**Proposition 3.7** *Let  $g \in W_0^N(\mathbb{R})$  have stable integer shifts, and let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity and height at most  $N < \infty$ . If  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^2(g)$ , then  $D^-(\Lambda, m_\Lambda) \geq 1$ .*

*A similar statement holds for the Paley–Wiener space  $\text{PW}^2$ .*

Proposition 3.7 follows from standard results on density of frames, see, e.g., [6,12]. See Sect. 7 for a sketch of a proof.



## 4 Density of Zero Sets in Shift-Invariant Spaces

We derive sharp upper bounds for the density of real zeros of functions in shift-invariant spaces with special generators. First, we use methods of complex analysis when the generator is a Gaussian (Sect. 4.1) or a hyperbolic secant (Sect. 4.2). The results and arguments are similar for both cases, but the case of the hyperbolic secant requires considerably more work and the analysis of meromorphic functions. In Sects. 4.3 and 4.4, we then analyze the zero sets in shift-invariant spaces generated by a totally positive function of Gaussian type by means of a comparison theorem.

### 4.1 The Gaussian

We now consider Gaussian functions  $\phi_a(x) := e^{-ax^2}$  with  $a > 0$ .

**Lemma 4.1** *Every  $f = \sum_k c_k \phi_a(\cdot - k) \in V^\infty(\phi_a)$  possesses an extension to an entire function satisfying the growth estimate*

$$|f(x + iy)| \leq C \|c\|_\infty e^{ay^2} \quad x, y \in \mathbb{R}. \quad (4.1)$$

**Proof** Using

$$e^{-a(x+iy-k)^2} = e^{ay^2} e^{-2aixy} e^{2aiky} e^{-a(x-k)^2},$$

we obtain that

$$f(x + iy) = e^{ay^2} e^{-2aixy} \sum_{k \in \mathbb{Z}} c_k e^{2aiky} e^{-a(x-k)^2}. \quad (4.2)$$

Consequently,

$$|f(x + iy)| \leq e^{ay^2} \|c\|_\infty \sum_{k \in \mathbb{Z}} e^{-a(x-k)^2},$$

and we may take  $C = \sup_{0 \leq x \leq 1} \sum_{k \in \mathbb{Z}} e^{-a(x-k)^2}$ . Clearly,  $x + iy \mapsto f(x + iy)$  is an entire function.  $\square$

Our key observation relates the real zeros of  $f \in V^\infty(\phi_a)$  to the zeros of its analytic extension.

**Lemma 4.2** *Let  $f \in V^\infty(\phi_a)$  and  $\lambda \in \mathbb{R}$  be a zero of  $f$  with multiplicity  $m$ . Then for every  $l \in \frac{\pi}{a}\mathbb{Z}$ ,  $\lambda + il$  is a zero of the analytic extension of  $f$  with the same multiplicity  $m$ . In particular, if  $f^{(j)}(\lambda) = 0$  for  $j = 0, \dots, m-1$ , then  $f^{(j)}(\lambda + il) = 0$  for  $j = 0, \dots, m-1$  and all  $l \in \frac{\pi}{a}\mathbb{Z}$ .*

**Proof** By (4.2) we obtain that

$$f(\lambda + il) = e^{al^2} e^{-2ai\lambda l} \sum_{k \in \mathbb{Z}} c_k e^{2aikl} e^{-a(\lambda-k)^2} = e^{al^2} e^{-2ai\lambda l} f(\lambda) = 0,$$

because  $e^{2aikl} = 1$  for all  $l \in \frac{\pi}{a}\mathbb{Z}$ .

For higher multiplicities we argue as follows. Note first that  $\frac{d^j}{dx^j}(e^{-ax^2}) = p_j(x)e^{-ax^2}$  for a polynomial of degree  $j$  satisfying the recurrence relation  $p_{j+1}(x) = -2axp_j(x) + p'_j(x)$ . It follows that the set  $\{p_j : j = 0, \dots, m-1\}$  is a basis for the polynomials of degree smaller than  $m$ .

Now assume that  $f \in V^\infty(\phi_a)$  and  $f^{(j)}(\lambda) = 0$  for  $j = 0, \dots, m-1$ . Then

$$\sum_{k \in \mathbb{Z}} c_k p_j(\lambda - k) e^{-a(\lambda - k)^2} = 0 \quad \text{for } j = 0, \dots, m-1.$$

This implies that for every polynomial  $q$  of degree  $< m$ ,

$$\sum_{k \in \mathbb{Z}} c_k q(\lambda - k) e^{-a(\lambda - k)^2} = 0. \quad (4.3)$$

We now proceed as in (4.2) and find that, for  $j = 0, \dots, m-1$ ,

$$\begin{aligned} f^{(j)}(\lambda + il) &= \sum_{k \in \mathbb{Z}} c_k p_j(\lambda - k + il) e^{-a(\lambda - k + il)^2} \\ &= e^{al^2} e^{-2ai\lambda l} \sum_{k \in \mathbb{Z}} c_k p_j(\lambda - k + il) e^{2aikl} e^{-a(\lambda - k)^2}. \end{aligned}$$

Note that  $e^{2aikl} = 1$  for all  $l \in \frac{\pi}{a}\mathbb{Z}$ . We insert the Taylor expansion of  $p_j$  at  $\lambda - k$ , i.e.,

$$p_j(\lambda - k + il) = \sum_{r=0}^j p_j^{(r)}(\lambda - k) \frac{(il)^r}{r!},$$

and we obtain that

$$f^{(j)}(\lambda + il) = e^{al^2} e^{-2ai\lambda l} \sum_{r=0}^j \frac{(il)^r}{r!} \sum_{k \in \mathbb{Z}} c_k p_j^{(r)}(\lambda - k) e^{-a(\lambda - k)^2}.$$

Since each  $p_j^{(r)}$  is a polynomial of degree  $< m$ , (4.3) implies that  $f^{(j)}(\lambda + il) = 0$  for all  $l \in \frac{\pi}{a}\mathbb{Z}$  and  $j = 0, \dots, m-1$ . This shows that the multiplicity of  $\lambda + il$  is at least that of  $\lambda$ . Reversing the roles of  $\lambda$  and  $\lambda + il$  we see that the multiplicities are actually equal.  $\square$

We recall Jensen's formula, which relates the number of zeros  $n(r)$  in a disk  $B(0, r)$  to the growth of an entire function by the identity

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|. \quad (4.4)$$

This is our main tool (from complex analysis) to prove the following result about the density of real zeros of functions in a shift-invariant space.

**Theorem 4.3** *Let  $\phi_a(x) = e^{-ax^2}$  with  $a > 0$ . Let  $f \in V^\infty(\phi_a) \setminus \{0\}$  and  $N_f$  its set of real zeros, with multiplicities  $m_f(x)$ ,  $x \in N_f$ . Then  $D^-(N_f, m_f) \leq 1$ .*

**Proof** Note that  $N_f = \{\lambda \in \mathbb{R} : f(\lambda) = 0\}$  is the set of *real* zeros of  $f$ . By Lemma 4.2, the set of *complex* zeros of (the analytic extension of)  $f$  contains the set  $N_f + i\frac{\pi}{a}\mathbb{Z} \subseteq \mathbb{C}$ , and, moreover, multiplicities are preserved.

To prove the theorem, we argue indirectly and assume that  $D^-(N_f, m_f) > 1$ . Then there exists  $\nu > 1$  and  $R_0$ , such that

$$\sum_{\lambda \in N_f \cap [x, x+r]} m_f(\lambda) \geq \nu r \quad \text{for all } x \in \mathbb{R}, r \geq R_0.$$

Let  $n(r)$  be the number of complex zeros of  $f$  inside the open disk  $B(0, r) \subseteq \mathbb{C}$  counted with multiplicities. Let us assume for the moment that  $f(0) \neq 0$ .

The right-hand side of Jensen's formula (4.4) can be estimated, by means of the growth estimate (4.1), as

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)| \leq A + \frac{1}{2\pi} \int_0^{2\pi} aR^2 \sin^2 \theta d\theta = A + \frac{aR^2}{2}, \quad (4.5)$$

where  $A := -\log|f(0)| + \log(\|c\|_\infty C)$ .

To estimate the left-hand side of (4.4), we choose  $R \in \mathbb{N}$  and  $R \geq R_0$  and partition  $[-R^2, R^2] = \bigcup_{k=-R}^{R-1} [kR, (k+1)R)$ . On each interval there are at least  $\nu R$  real zeros of  $f$  counted with multiplicity. By symmetry it is enough to consider intervals  $[kR, (k+1)R)$  with  $0 \leq k \leq R-1$ . By Lemma 4.2, for each real zero  $\lambda \in [kR, (k+1)R)$ , with a certain multiplicity  $m$ , there are  $2\lfloor \frac{a}{\pi} \sqrt{(R^2)^2 - \lambda^2} \rfloor + 1 \geq \frac{2a}{\pi} \sqrt{R^4 - (k+1)^2 R^2} - 1$  complex zeros  $\lambda + il$ ,  $l \in \frac{\pi}{a}\mathbb{Z}$  in the disk  $B(0, R^2)$ , each with multiplicity  $m$ . By counting with multiplicities, there are at least

$$\nu R \left( \frac{2a}{\pi} \sqrt{R^4 - (k+1)^2 R^2} - 1 \right)$$

complex zeros in  $B(0, R^2)$  with real part in  $[kR, (k+1)R)$ , where  $0 \leq k \leq R-1$ . By summing over (positive and negative)  $k$ , we obtain the following lower bound for the number of complex zeros of  $f$  in  $B(0, R^2)$ :

$$\begin{aligned} n(R^2) &\geq 2\nu R \sum_{k=0}^{R-1} \left( \frac{2a}{\pi} \sqrt{R^4 - (k+1)^2 R^2} - 1 \right) \\ &= \frac{4\nu a R^4}{\pi} \sum_{k=0}^{R-1} \frac{1}{R} \sqrt{1 - \frac{(k+1)^2}{R^2}} - 2\nu R^2. \end{aligned}$$

The last sum is a Riemann sum of the integral  $\int_0^1 \sqrt{1-x^2} dx = \pi/4$ . Let  $\epsilon > 0$  and  $R_1 \geq R_0$  satisfy  $\beta := v(1 - \epsilon - \frac{2}{aR_1^2}) > 1$ . Then, for some  $R_2 \geq R_1$  and all  $R \geq R_2$ , we conclude that

$$n(R^2) \geq avR^4 \left(1 - \epsilon - \frac{2}{aR^2}\right) \geq a\beta R^4,$$

or, equivalently,

$$n(r) \geq a\beta r^2 \text{ for } r \geq R_2^2.$$

Therefore, the left-hand side of (4.4) can be estimated as

$$\int_0^R \frac{n(r)}{r} dr \geq \int_{R_2^2}^R \frac{n(r)}{r} dr \geq a\beta \left( \frac{R^2}{2} - \frac{R_2^2}{2} \right).$$

Since  $\beta > 1$ , this estimate is incompatible with the growth of  $f$  as encoded in (4.5). Therefore  $D^-(N_f, m_f) > 1$  is impossible. This concludes the proof for  $f$  such that  $f(0) \neq 0$ .

If  $f(0) = 0$ , we let  $n \geq 0$  be the vanishing order of  $f$  at 0 and apply the previous argument to  $\tilde{f}(z) := z^{-n}f(z)$ . Alternatively, one can verify directly that if  $f \not\equiv 0$ , then there exists  $k \in \mathbb{Z}$  such that  $f(k) \neq 0$ , and consider  $\tilde{f}(x) = f(x+k)$  with  $\tilde{f} \in V^\infty(\phi)$ .  $\square$

## 4.2 The Hyperbolic Secant

Let  $\psi_a(x) = \operatorname{sech}(ax) = \frac{2}{e^{ax} + e^{-ax}}$ . Our goal is to study the shift-invariant space generated by  $\psi_a$ . While in [15] we studied  $V^2(\psi_a)$  by exploiting a connection to Gabor analysis, and a certain representation of the Zak transform of  $\psi_a$  due to Janssen and Strohmer [17], here we consider meromorphic extensions of the functions in  $V^\infty(\psi_a)$ .

We introduce the following notation. For real  $x$  we denote the roundoff error to the nearest integer as  $\{x\} := x - l$ , where  $l \in \mathbb{Z}$  and  $\{x\} \in [-1/2, 1/2)$ .

**Lemma 4.4** *Every  $f = \sum_{k \in \mathbb{Z}} c_k \psi_a(\cdot - k) \in V^\infty(\psi_a)$  has an extension to a meromorphic function on  $\mathbb{C}$  with poles in*

$$P_f \subseteq P := \mathbb{Z} + \frac{i\pi}{a} \left( \frac{1}{2} + \mathbb{Z} \right).$$

Moreover, every pole of  $f$  is simple, and  $f$  satisfies the growth estimate

$$\begin{aligned} |f(x + iy)| &\leq C \|c\|_\infty |\psi_a(\{x\} + iy)| \\ &\leq C \|c\|_\infty \min\{|a\{x\}|^{-1}, |2\{\frac{ay}{\pi} - \frac{1}{2}\}|^{-1}\}. \end{aligned} \quad (4.6)$$

**Proof** The meromorphic function  $\psi_a(z) = \operatorname{sech}(az)$  has simple poles on the imaginary axis at  $\frac{i\pi}{a}(\frac{1}{2} + \mathbb{Z})$ . The identity

$$|\cosh a(x - k + iy)| = (\sinh^2 a(x - k) + \cos^2 ay)^{1/2}$$

shows that  $|\psi_a(x - k + iy)| \lesssim e^{-a|x-k|}$ , if  $|x - k| \geq 1$  and  $y$  is arbitrary. We consider the covering of  $\mathbb{C}$  given by

$$U_{s,t} := \left\{ x + iy \in \mathbb{C} : |x - s| < 3/4, |y - \frac{\pi}{a}(t + 1/2)| < \frac{3\pi}{4a} \right\}, \quad s, t \in \mathbb{Z}.$$

On  $U_{s,t}$ , the partial sums

$$f_N(x + iy) = \sum_{k:|k-s| \leq N} c_k \psi_a(x - k + iy)$$

have at most a simple pole at  $s + \frac{i\pi}{a}(\frac{1}{2} + t)$  and are otherwise analytic. Since, for  $x + iy \in U_{s,t}$ ,

$$\sum_{k:|k-s| > N} |c_k| |\psi_a(x - k + iy)| \lesssim \|c\|_\infty \sum_{k:|k-s| > N} e^{-a|x-k|} \lesssim \|c\|_\infty e^{-N},$$

the partial sums  $f_N$  converge uniformly on  $U_{s,t} \setminus \{s + \frac{i\pi}{a}(\frac{1}{2} + t)\}$  to an analytic extension of  $f$ . More precisely,

$$\sup_{z \in U_{s,t} \setminus \{s + \frac{i\pi}{a}(1/2+t)\}} |f(z) - f_N(z)| \longrightarrow 0, \quad \text{as } N \longrightarrow \infty.$$

(Note that this is stronger than the usual uniform convergence on compact sets.) This fact implies that  $f$  has at most a simple pole at  $z = s + \frac{i\pi}{a}(1/2 + t)$ . Hence,  $f$  is meromorphic on  $\mathbb{C}$  with at most simple poles in  $\mathbb{Z} + \frac{i\pi}{a}(\frac{1}{2} + \mathbb{Z})$ .

For the growth estimate (4.6), we let  $x + iy \in \mathbb{C} \setminus P_f$  and write  $x = l + \{x\}$  with  $l \in \mathbb{Z}$  and  $\{x\} \in [-1/2, 1/2)$ . Then we have

$$|f(x + iy)| \leq |\psi_a(\{x\} + iy)| \left( |c_l| + \sum_{k \neq l} \left| \frac{c_k \psi_a(x - k + iy)}{\psi_a(\{x\} + iy)} \right| \right).$$

For all  $k \neq l$ , we observe that  $|x - k| \geq |l - k| - |\{x\}| \geq \frac{1}{2} \geq |\{x\}|$ . Therefore, we have  $\sinh^2 a(x - k) \geq \sinh^2 a\{x\}$ . Since the rational function  $r(y) = \frac{c+y}{d+y}$  with  $d \geq c \geq 0, d > 0$ , is increasing for  $y > 0$ , we obtain that, for all  $k \neq l$ ,

$$\begin{aligned} \frac{|\psi_a(x - k + iy)|^2}{|\psi_a(\{x\} + iy)|^2} &= \frac{\sinh^2 a\{x\} + \cos^2 ay}{\sinh^2 a(x - k) + \cos^2 ay} \leq \frac{\sinh^2 a\{x\} + 1}{\sinh^2 a(x - k) + 1} \\ &= \frac{\cosh^2 a\{x\}}{\cosh^2 a(x - k)}, \end{aligned}$$

and furthermore

$$\frac{\cosh a \{x\}}{\cosh a(x-k)} = \frac{e^{a\{x\}}(1 + e^{-2a\{x\}})}{e^{a|x-k|}(1 + e^{-2a|x-k|})} \leq 2e^a e^{-a|k-l|}.$$

Therefore, we have

$$|c_l| + \sum_{k \neq l} \left| \frac{c_k \psi_a(x-k+iy)}{\psi_a(\{x\}+iy)} \right| \leq \|c\|_\infty \left( 1 + 2e^a \sum_{k \neq l} e^{-a|k-l|} \right) \leq C \|c\|_\infty.$$

This proves the first inequality in (4.6). For the second inequality, note that

$$|\sinh ax| \geq |ax| \quad \text{for all } x \in \mathbb{R},$$

and, by periodicity and elementary trigonometric identities,

$$|\cos ay| = |\sin \pi \left\{ \frac{ay}{\pi} - \frac{1}{2} \right\}| \geq 2 \left| \left\{ \frac{ay}{\pi} - \frac{1}{2} \right\} \right| \quad \text{for all } y \in \mathbb{R}.$$

Hence, we obtain

$$|\psi_a(\{x\}+iy)| = \left( \sinh^2 a \{x\} + \cos^2 ay \right)^{-1/2} \leq \min\{|a \{x\}|^{-1}, |2 \left\{ \frac{ay}{\pi} - \frac{1}{2} \right\}|^{-1}\},$$

which gives the second inequality in (4.6).  $\square$

The following result is an analogue of Lemma 4.2 for  $V^\infty(\psi_a)$ .

**Lemma 4.5** *Let  $f \in V^\infty(\psi_a)$  and  $\lambda \in \mathbb{R}$  be a zero of  $f$  with multiplicity  $m$ . Then for every  $l \in \frac{\pi}{a}\mathbb{Z}$ ,  $\lambda + il$  is a zero of the meromorphic extension of  $f$  with the same multiplicity  $m$ .*

**Proof** For every  $x \in \mathbb{R}$  and  $l = \frac{\pi t}{a} \in \frac{\pi}{a}\mathbb{Z}$ , we have

$$\cosh a(x+il) = \cosh ax \cos al + i \sinh ax \sin al = (-1)^t \cosh ax.$$

Therefore, every  $f = \sum_{k \in \mathbb{Z}} c_k \psi_a(\cdot - k) \in V^\infty(\psi_a)$  satisfies

$$f(x+il) = \sum_k c_k (-1)^t \psi_a(x-k) = (-1)^t f(x).$$

This implies that the Taylor expansions of  $f$  around  $z_0 = x \in \mathbb{R}$  and around  $z_l = x + il$  have exactly the same coefficients, up to a factor  $(-1)^t$ . In particular,  $f^{(j)}(\lambda) = 0$  holds for some  $\lambda \in \mathbb{R}$  and  $j \geq 0$  if and only if  $f^{(j)}(\lambda + il) = 0$  for all  $l \in \frac{\pi}{a}\mathbb{Z}$ .  $\square$

Let  $n(r)$  denote the difference of the number of zeros and the number of poles of  $f$  in the closed disk  $\overline{B(0, r)}$ , counted with multiplicities. Jensen's formula for meromorphic functions  $f$  with  $f(0) \notin \{0, \infty\}$  says that

$$\int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta - \log|f(0)|,$$

see, e.g., [16, pages 4–6]. The special case  $f(0) = 0$  or  $\infty$  is treated as follows: if  $f$  has a zero or pole at 0, choose  $m \in \mathbb{Z}$  such that  $\lim_{z \rightarrow 0} f(z)/z^m = C_m \neq 0$ . Then

$$\int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta - \log|C_m| - m \log r. \quad (4.7)$$

After this excursion to meromorphic functions we can now prove an analogue of Theorem 4.3.

**Theorem 4.6** *Let  $f \in V^\infty(\psi_a) \setminus \{0\}$  and  $N_f$  its set of real zeros with multiplicities  $m_f$ . Then  $D^-(N_f, m_f) \leq 1$ .*

The main part of the proof is an estimate of the integral in Jensen's formula.

**Lemma 4.7** *For every  $f \in V^\infty(\psi_a)$ , we have*

$$\sup_{r>1} \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta < \infty.$$

**Proof** We divide the integral into four pieces corresponding to

$$\theta \in I_j = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] + \frac{j\pi}{2}, \quad j = 0, 1, 2, 3.$$

For  $\theta \in I_0 \cup I_2$ , we let

$$re^{i\theta} = \pm\sqrt{r^2 - y^2} + iy, \quad \text{where } y \in \left[-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right].$$

By (4.6), we have

$$\log|f(re^{i\theta})| \leq \log(C\|c\|_\infty) - \log 2 \left\{ \frac{ay}{\pi} - \frac{1}{2} \right\}|$$

and (using  $d\theta = \pm dy/\sqrt{r^2 - y^2}$ )

$$\frac{1}{2\pi} \int_{I_0 \cup I_2} \log|f(re^{i\theta})| d\theta \leq \frac{1}{2} \log(C\|c\|_\infty) - \frac{1}{\pi} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \log 2 \left\{ \frac{ay}{\pi} - \frac{1}{2} \right\}| \frac{dy}{\sqrt{r^2 - y^2}}.$$

Note that  $\log|2\{\frac{ay}{\pi} - \frac{1}{2}\}| \leq 0$  and  $\sqrt{r^2 - y^2} \geq r/\sqrt{2}$  for all  $y \in [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}]$ . Therefore,

$$-\frac{1}{\pi} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \log|2\{\frac{ay}{\pi} - \frac{1}{2}\}| \frac{dy}{\sqrt{r^2 - y^2}} \leq \frac{\sqrt{2}}{\pi r} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \left| \log|2\{\frac{ay}{\pi} - \frac{1}{2}\}| \right| dy.$$

For the last integral, we use the substitution  $u = \frac{ay}{\pi}$  and observe that the resulting integrand is even and periodic with period 1. This gives for all  $c < d$ ,

$$\begin{aligned} \int_c^d |\log|2\{\frac{ay}{\pi} - \frac{1}{2}\}|| dy &= \frac{\pi}{a} \int_{\frac{ac}{\pi}}^{\frac{ad}{\pi}} |\log|2\{u - \frac{1}{2}\}|| du \\ &\leq \frac{2\pi}{a} \left( \frac{ad}{\pi} - \frac{ac}{\pi} + 1 \right) \int_0^{1/2} |\log(2u)| du = d - c + \frac{\pi}{a}, \end{aligned}$$

and finally

$$\frac{1}{2\pi} \int_{I_0 \cup I_2} \log|f(re^{i\theta})| d\theta \leq \frac{1}{2} \log(C\|c\|_\infty) + \frac{\sqrt{2}}{\pi r} \left( \sqrt{2}r + \frac{\pi}{a} \right).$$

In the same way, for  $\theta \in I_1 \cup I_3$ , we let

$$re^{i\theta} = x \pm i\sqrt{r^2 - x^2}, \quad \text{where } x \in \left[-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right],$$

and obtain from (4.6)

$$\frac{1}{2\pi} \int_{I_1 \cup I_3} \log|f(re^{i\theta})| d\theta \leq \frac{1}{2} \log(C\|c\|_\infty) - \frac{1}{\pi} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \log|a\{x\}| \frac{dx}{\sqrt{r^2 - x^2}}.$$

The same techniques as before give

$$-\frac{1}{\pi} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \log|a\{x\}| \frac{dx}{\sqrt{r^2 - x^2}} \leq \frac{\sqrt{2}}{\pi r} \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \left| \log|a\{x\}| \right| dx,$$

and, for every  $d > c$ ,

$$\begin{aligned} \int_c^d |\log|a\{x\}|| dx &\leq (d - c)|\log(a/2)| + \int_c^d |\log|2\{x\}|| dx \\ &\leq (d - c)(|\log(a/2)| + 2(d - c + 1) \int_0^{1/2} |\log(2u)| du \\ &\leq (d - c + 1)(1 + |\log(a/2)|). \end{aligned}$$



Hence, we obtain

$$\frac{1}{2\pi} \int_{I_1 \cup I_3} \log |f(re^{i\theta})| d\theta \leq \frac{1}{2} \log(C\|c\|_\infty) + \frac{\sqrt{2}}{\pi r} (\sqrt{2}r + 1) (1 + |\log(a/2)|).$$

Combining both integrals, we get for  $r \geq 1$ ,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta &\leq \log(C\|c\|_\infty) + \frac{\sqrt{2}}{\pi r} (2\sqrt{2}r + 1 + \frac{\pi}{a} \\ &\quad + (\sqrt{2}r + 1)|\log(a/2)|), \end{aligned}$$

which is bounded for  $r \geq 1$ . This completes the proof.  $\square$

**Proof of Theorem 4.6** Assume  $D^-(N_f) > 1$ . Let  $n_z(r)$  denote the number of zeros of  $f$  in  $\overline{B(0, r)}$  and  $n_p(r)$  the number of poles in that disk (both counted with multiplicities). The same counting argument as in the proof of Theorem 4.3, this time invoking Lemma 4.5, gives

$$n_z(r) \geq a\beta r^2$$

for some  $\beta > 1$  and for all  $r \geq R_1$ . On the other hand, by Lemma 4.4, the poles of  $f$  are simple and contained in the shifted lattice  $\mathcal{L} := \mathbb{Z} + \frac{i\pi}{a}(\frac{1}{2} + \mathbb{Z})$ . To find an upper bound for  $n_p(r)$ , we place rectangles  $Q_x = x + [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{\pi}{2a}, \frac{\pi}{2a}]$  of area  $|Q_x| = \frac{\pi}{a}$  and diagonal  $\sqrt{1 + \pi^2/a^2}$  around each pole and observe that

$$\begin{aligned} n_p(r) &= \frac{a}{\pi} \left| \bigcup_{x \in \mathcal{L} \cap \overline{B(0, r)}} Q_x \right| \leq \frac{a}{\pi} \left| B\left(0, r + \frac{1}{2}\sqrt{1 + \pi^2/a^2}\right) \right| \\ &= a \left( r + \frac{1}{2}\sqrt{1 + \pi^2/a^2} \right)^2. \end{aligned}$$

As a consequence,  $n(r) = n_z(r) - n_p(r) \geq (\beta - 1)ar^2 - cr$ , for some constant  $c > 0$ , and

$$\int_{R_1}^R \frac{n(r)}{r} dr \geq \frac{(\beta - 1)a(R^2 - R_1^2)}{2} - c(R - R_1).$$

Due to Lemma 4.7, for  $R \gg 1$  this contradicts Jensen's formula (4.7).  $\square$

### 4.3 Transference of Zero Sets

The following lemma modifies [15, Lemma 5.1] to include multiplicities and allows us to compare the density of zero sets in different shift-invariant spaces.

**Lemma 4.8** *Let  $f \in C^\infty(\mathbb{R})$  be real-valued and  $m_f : N_f \rightarrow \mathbb{N}$  be the multiplicity function of its zeros. For  $a \in \mathbb{R}$ , let  $g = (aI + \frac{d}{dx})f$ . Then*

$$D^-(N_g, m_g) \geq D^-(N_f, m_f). \quad (4.8)$$

*For  $f \in C^{N-1}(\mathbb{R})$  the same statement holds, replacing  $m_f$  and  $m_g$  by the multiplicity functions of the zeros of height at most  $N$  and  $N - 1$ , respectively.*

**Proof** Let  $f \in C^\infty(\mathbb{R})$ . Note that  $aI + \frac{d}{dx} = e^{-ax} \frac{d}{dx} e^{ax}$ . We define  $h \in C^\infty(\mathbb{R})$  by  $h(x) = e^{ax} f(x)$  and note that  $N_h = N_f$  with equal multiplicities  $m_h = m_f$ . Furthermore, since

$$g(x) = \left(aI + \frac{d}{dx}\right)f(x) = e^{-ax} h'(x),$$

we conclude that  $N_g = N_{h'}$ , again with equal multiplicities  $m_g = m_{h'}$ . It remains to show that  $D^-(N_{h'}, m_{h'}) \geq D^-(N_h, m_h)$ .

Let  $x \in \mathbb{R}$ ,  $R > 0$ , and  $F \subseteq N_h \cap [x - R, x + R]$  a finite subset. All zeros  $x$  of  $h$  with multiplicity  $m_h(x) > 1$  are zeros of  $h'$  with multiplicity  $m_{h'}(x) = m_h(x) - 1$ . Since the zeros of  $h$  and  $h'$  are interlaced by Rolle's theorem, we obtain additional zeros  $\tilde{F} \subset [x - R, x + R] \setminus F$  of  $h'$  with cardinality  $\#\tilde{F} = \#F - 1$ . Combining both types of zeros of  $h'$  gives

$$\sum_{x \in F \cup \tilde{F}} m_{h'}(x) \geq \left( \sum_{x \in F} m_h(x) \right) - 1.$$

Since this holds for every finite subset  $F \subseteq N_h \cap [x - R, x + R]$ , it follows that  $D^-(N_{h'}, m_{h'}) \geq D^-(N_h, m_h)$ , and

$$D^-(N_g, m_g) = D^-(N_{h'}, m_{h'}) \geq D^-(N_h, m_h) = D^-(N_f, m_f),$$

as claimed. Finally, for  $f \in C^{N-1}(\mathbb{R})$ ,  $g \in C^{N-2}(\mathbb{R})$ , and the same argument applies to the multiplicity functions of zeros of height  $N$  and  $N - 1$ .  $\square$

Although generically one would expect equality in (4.8), the density of the zero set may actually jump. Let  $f(x) = \sum_{k \in \mathbb{Z}} e^{-\pi(x-k)^2} \in V^\infty(\phi_\pi)$ , and  $h = f' \in V^\infty(\phi'_\pi)$ . Then  $f$  is a nonconstant, strictly positive, periodic, real-valued function, and we have  $N_f = \emptyset$  and  $D^-(N_f) = 0$ . Since  $f$  assumes two extremal values in  $[0, 1)$ , we have  $D^-(N_h) = 2$ , in fact,  $N_h = \frac{1}{2}\mathbb{Z}$ . This example explains why the methods of this paper cannot be applied directly to sampling in shift-invariant spaces generated by Hermite functions. Indeed, Theorem 1.1 does not have a direct analog for  $V(h_n)$  with the  $n$ -th Hermite function  $h_n$ ,  $n > 0$ .

#### 4.4 Totally Positive Functions of Gaussian Type

We next study shift-invariant spaces generated by a totally positive function of Gaussian type and their density of zeros.

**Theorem 4.9** *Let  $g$  be a totally positive function of Gaussian type. Let  $f \in V^\infty(g) \setminus \{0\}$  be real-valued and  $(N_f, m_f)$  its set of real zeros counted with multiplicities. Then  $D^-(N_f, m_f) \leq 1$ .*

*In particular, if  $D^-(\Lambda) > 1$ , then  $\Lambda$  is a uniqueness set for  $V^\infty(g)$ .*

**Proof** The proof is an adaption of the argument in [15] using multiplicities. Recall that  $g$  is real-valued and has stable integer shifts. Let  $c \in \ell^\infty(\mathbb{Z})$ , and assume that  $f = \sum_{k \in \mathbb{Z}} c_k g(\cdot - k) \in V^\infty(g)$  vanishes on  $N_f \subset \mathbb{R}$  with  $D^-(N_f, m_f) > 1$ . We want to show that  $f \equiv 0$ . Note that  $f \in C^\infty(\mathbb{R})$ . Since  $g$  is real-valued, we may assume without loss of generality that  $f$  is also real-valued (by replacing  $c_k$  by  $\Re(c_k)$  or  $\Im(c_k)$  if necessary).

Using (1.1), write

$$\hat{g}(\xi) = \prod_{j=1}^n (1 + 2\pi i \delta_j \xi)^{-1} \hat{\phi}(\xi), \quad \delta_1, \dots, \delta_n \in \mathbb{R} \setminus \{0\}, \quad c > 0,$$

where  $\hat{\phi}(\xi) = e^{-c\xi^2}$ . In other words,  $\phi = \prod_{j=1}^n (I + \delta_j \frac{d}{dx}) g$  is a Gaussian. Since  $\phi, g$ , and their derivatives decay exponentially, we may interchange summation and differentiation in  $f$ , and obtain that

$$h = \prod_{j=1}^n \left( I + \delta_j \frac{d}{dx} \right) f \in V^\infty(\phi).$$

The repeated use of Lemma 4.8 implies that  $D^-(N_h, m_h) \geq D^-(N_f, m_f) > 1$ . Hence, by Theorem 4.3,  $h = \sum_k c_k \phi(\cdot - k) \equiv 0$ . Hence  $c_k \equiv 0$  and  $f \equiv 0$ , as claimed.  $\square$

#### 4.5 Bandlimited Functions

For a simple comparison of the results in Theorems 4.3, 4.6, and 4.9, we mention the following result for bandlimited functions.

**Theorem 4.10** *Let  $f \in \text{PW}^\infty \setminus \{0\}$ . Then  $D^-(N_f, m_f) \leq 1$ .*

**Proof** The result follows from the Paley–Wiener characterization of bandlimited functions as restrictions of entire functions of exponential type, and Jensen’s formula. Beurling’s proof [7,8] applies almost verbatim.  $\square$

## 5 Proof of the Sampling Theorems

The proofs of our main theorems are now short and follow from the combination of the characterization of sampling sets without inequalities (Theorem 3.4) and the new insights about the density of zero sets in shift-invariant spaces (Sect. 4).

**Proof of Theorem 1.1** The necessity of the density conditions is stated in Proposition 3.7. For the sufficiency, we apply the characterization of Theorem 3.4. Suppose that  $D^-(\Lambda, m_\Lambda) > 1$ , and let  $(\Gamma, m_\Gamma) \in W_{\mathbb{Z}}(\Lambda, m_\Lambda)$ . By Lemma 3.3,  $D^-(\Gamma, m_\Gamma) > 1$ . Hence, by Theorem 4.9,  $(\Gamma, m_\Gamma)$  is a uniqueness set for  $V^\infty(g)$ . Therefore, the criterion in Theorem 3.4 is satisfied, and we conclude that  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^2(g)$ .  $\square$

**Proof of Theorem 1.2** The proof is the same as for Theorem 1.1; this time we resort to Theorem 4.6 (instead of Theorem 4.9).  $\square$

**Proof of Theorem 1.3** The first part of the proof (treating  $PW^\infty$ ) is similar to the one of Theorem 1.1. If  $(\Lambda, m_\Lambda)$  is a separated set with finite height and density  $D^-(\Lambda, m_\Lambda) > 1$ , then Theorem 3.5 (combined with Lemma 3.3 and Theorem 4.10) implies that  $(\Lambda, m_\Lambda)$  is a sampling set for  $PW^\infty$ . As a second step, we use Proposition 3.6 to extend the conclusion to  $PW^2$ . More precisely, if  $D^-(\Lambda, m_\Lambda) > 1$ , we select  $\alpha < 1$  such that  $\alpha D^-(\Lambda, m_\Lambda) = D^-(\alpha^{-1}\Lambda, m_\Lambda) > 1$ . We conclude that  $(\alpha^{-1}\Lambda, m_\Lambda)$  is a sampling set for  $PW^\infty$ , and therefore, by Proposition 3.6,  $(\Lambda, m_\Lambda)$  is a sampling set for  $PW^2$ .  $\square$

## 6 Consequences for Gabor Frames

The Hermite-sampling results of Theorems 1.1 and 1.2 can be applied in order to obtain sharp density results for multi-window Gabor frames. This extends our previous work in [15] and was, in fact, one of our original motivations for the present work. We obtain new families of multi-window Gabor frames with optimal conditions for semi-regular sets of time-frequency shifts.

### 6.1 Multi-window Gabor Frames

Let  $\pi(x, w)g(t) = g(t - x)e^{2\pi i w t}$  denote the time-frequency shift of  $g$  by  $(x, w) \in \mathbb{R} \times \mathbb{R}$ . For given windows  $g^1, \dots, g^N \in L^2(\mathbb{R})$  and sets  $\Delta^1, \dots, \Delta^N \subseteq \mathbb{R}^2$ , the associated multi-window Gabor system is

$$\mathcal{G}(g^1, \dots, g^N, \Delta^1, \dots, \Delta^N) = \left\{ \pi(x, w)g^j : (x, w) \in \Delta^j, j = 1, \dots, N \right\}. \quad (6.1)$$

It will be convenient to use the notation  $G = (g^1, \dots, g^N)$ ,  $\vec{\Delta} = (\Delta^1, \dots, \Delta^N)$ , and  $\mathcal{G}(G, \vec{\Delta})$ . When all the sets  $\Delta^j$  are equal, we just write  $\mathcal{G}(G, \Delta)$ .

## 6.2 Connection Between Sampling and Gabor Frames

For semi-regular sets  $\vec{\Delta}$ , the Gabor frame property can be related to a sampling problem as follows.

**Theorem 6.1** *Assume that  $G = (g^1, \dots, g^N) \in (W_0(\mathbb{R}))^N$  has stable integer shifts and that the sets  $\Lambda^1, \dots, \Lambda^N \subseteq \mathbb{R}$  are separated. Let  $\vec{\Delta} = (\Delta^1, \dots, \Delta^N)$  be given by  $\Delta^j := (-\Lambda^j) \times \mathbb{Z}$ .*

*Then  $\mathcal{G}(G, \vec{\Delta})$  is a frame for  $L^2(\mathbb{R})$  if and only if  $\vec{\Lambda} + (x, \dots, x)$  is a sampling set for  $V^2(G)$  for all  $x \in \mathbb{R}$ .*

Theorem 6.1 is a vector-valued extension of [15, Theorem 2.3] (equivalence of conditions (a) and (b)), and we therefore omit its proof.

## 6.3 Characterization of Multi-window Gabor Frames with Totally Positive Windows

**Theorem 6.2** *Let  $g$  be a totally positive function of Gaussian-type or the hyperbolic secant, and let  $\Lambda \subseteq \mathbb{R}$  be a separated set. Let  $\{p_1, \dots, p_N\}$  be a basis of the space of polynomials of degree less than  $N$ . Set  $g^j = p_j(\frac{d}{dx})g$  and  $G = (g^1, \dots, g^N)$ .*

*Then  $\mathcal{G}(G, (-\Lambda) \times \mathbb{Z})$  is a frame for  $L^2(\mathbb{R})$  if and only if  $D^-(\Lambda) > 1/N$ .*

**Proof** The necessity of the condition  $D^-(\Lambda) > 1/N$  for multi-window Gabor frames over a rectangular lattice is contained in [26, Thm. 12.2.11]. It also follows from general results, see, e.g., [13].

For the sufficiency, assume that  $D^-(\Lambda) > 1/N$ , and let  $\vec{\Lambda} := (\Lambda, \dots, \Lambda)$ . By Theorem 6.1, it suffices to show that for all  $x \in \mathbb{R}$ ,  $\vec{\Lambda} + \vec{x}$  is a sampling set for the vector-valued shift-invariant space  $V^2(G)$ , where  $\vec{x} := (x, \dots, x)$ . To verify this condition, we apply Theorem 3.4.

Let  $\vec{\Gamma} \in W_{\mathbb{Z}}(\vec{\Lambda} + \vec{x})$ . This set is necessarily of the form  $\vec{\Gamma} = (\Gamma, \dots, \Gamma)$ , for some  $\Gamma \in W_{\mathbb{Z}}(\Lambda + x)$ , and, by Lemma 3.3,  $D^-(\Gamma) > 1/N$ . Assume that  $F \in V^\infty(G)$  vanishes on  $\vec{\Gamma}$ . We need to show that  $F \equiv 0$ . Explicitly  $F$  is given by an expansion  $F = \sum_{k \in \mathbb{Z}} c_k G(\cdot - k)$  with  $c \in \ell^\infty(\mathbb{Z})$ .

We now relate the sampling problem for vector-valued functions to a sampling problem with derivatives. To do this, we set  $P = (p_1, \dots, p_N)$  and  $Q = (1, x, \dots, x^{N-1})$ . By assumption on  $P$ , there is an invertible  $N \times N$ -matrix  $B$ , such that  $BP = Q$ , i.e.,  $x^{j-1} = \sum_{k=1}^N b_{jk} p_k(x)$  for  $j = 1, \dots, N$ , and thus

$$\sum_{k=1}^N b_{jk} g^k = \sum_{k=1}^N b_{jk} p_k\left(\frac{d}{dx}\right)g = g^{(j-1)}.$$

Consequently, after taking linear combinations of translates, we obtain

$$BF(x)_j = \sum_{l \in \mathbb{Z}} c_l BG(x-l)_j = \sum_{l \in \mathbb{Z}} c_l g^{(j-1)}(x-l) = f^{(j-1)}(x),$$

where  $f = \sum_l c_l g(\cdot - l) \in V^\infty(g)$  is the first component of  $BF$ . If  $F$  vanishes on  $\vec{\Gamma}$ , then also  $f^{(j-1)}$  vanishes on  $\Gamma$  for  $j = 1, \dots, N$ . Hence,  $f$  vanishes on  $\Gamma$

with multiplicity  $N$  and  $D^-(N_f, m_f) \geq ND^-(\Gamma) > 1$ . By Theorem 4.9 or 4.6, this implies that  $f \equiv 0$ . Hence,  $c_k \equiv 0$  and  $F \equiv 0$ , as desired.  $\square$

We single out two special cases of Theorem 6.2.

**Corollary 6.3** *Let  $g$  be a totally positive function of Gaussian-type or the hyperbolic secant, and let  $\Lambda \subseteq \mathbb{R}$  be a separated set. Let  $a_1, a_2, \dots, a_{N-1} \in \mathbb{R}$ ,  $g^1 = g$ , and set*

$$g^j := \prod_{k=1}^{j-1} \left( a_k I + \frac{d}{dx} \right) g, \quad j = 2, \dots, N, \quad (6.2)$$

*and  $G = (g^1, \dots, g^N)$ . Then  $\mathcal{G}(G, \Lambda \times \mathbb{Z})$  is a frame for  $L^2(\mathbb{R})$  if and only if  $D^-(\Lambda) > 1/N$ .*

For the second corollary, we use the basis of Hermite functions  $\{h_k : k \geq 0\}$  which is defined by

$$h_k(x) = \gamma_k e^{\pi x^2} \frac{d^k}{dx^k} e^{-2\pi x^2} = (-1)^k \gamma_k e^{-\pi x^2} H_k(x),$$

with the Hermite polynomials  $H_k$  of degree  $k$  and some normalizing constant  $\gamma_k > 0$ .

**Corollary 6.4** *Let  $\Lambda \subseteq \mathbb{R}$  be a separated set and  $b > 0$ . Then  $\mathcal{G}(h_0, \dots, h_{N-1}, \Lambda \times b\mathbb{Z})$  is a frame of  $L^2(\mathbb{R})$  if and only if  $D^-(\Lambda) > b/N$ .*

**Proof** We use the fact that  $\mathcal{G}(h_0, \dots, h_{N-1}, \Lambda \times b\mathbb{Z})$  is a frame if and only if

$$\mathcal{G}(h_0(b^{-1}\cdot), \dots, h_{N-1}(b^{-1}\cdot), b\Lambda \times \mathbb{Z})$$

is. Because the Hermite polynomials  $H_k, k = 0, \dots, N-1$ , form a basis for the polynomials of degree  $< N$ , the span of  $h_k, 0 \leq k \leq N-1$ , is the same as the span of all derivatives  $\frac{d^j}{dx^j} e^{-\pi x^2}, 0 \leq j \leq N-1$ . The result is a consequence of Theorem 6.2.  $\square$

Corollary 6.4 actually follows from a sampling result of Brekke and Seip in Fock space [9]. It can also be reformulated for spaces of polyanalytic functions. For this connection see [1]. Related questions for wavelets, including the behavior of the frame bounds when the number of windows increases, have been investigated in [24].

## 7 Postponed Proofs

### 7.1 Proof of Proposition 3.2

For sets without multiplicities, i.e.,  $m_\Lambda \equiv 1$ , the proposition is classical.

Let  $(\Lambda, m_\Lambda)$  be a separated set with multiplicity with finite height, let  $(\Gamma, m_\Gamma)$  be a set with multiplicity, and  $\{k_n : n \geq 1\} \subseteq \mathbb{Z}$ . Recall that  $\Lambda^j = \{\lambda \in \Lambda : m(\lambda) \geq j\}$ .

Suppose first that  $\Lambda^j - k_n \xrightarrow{w} \Gamma^j$ , as  $n \rightarrow \infty$  for all  $j = 1, \dots, N$ . Then, by the case without multiplicity,  $\sum_{\lambda \in \Lambda^j} \delta_{\lambda - k_n} \rightarrow \sum_{\gamma \in \Gamma^j} \delta_\gamma$ , in the  $\sigma(C_c^*, C_c)$  topology. Since  $(\Lambda, m_\Lambda)$  has finite height, the claim follows by summing over  $j$ .

Conversely, assume that  $\mu_n := \sum_{\lambda \in \Lambda} m_\Lambda(\lambda) \delta_{\lambda - k_n} \rightarrow \mu := \sum_{\gamma \in \Gamma} m_\Gamma(\gamma) \delta_\gamma$ , in the  $\sigma(C_c^*, C_c)$  topology. As discussed in [13, Lemmas 4.3, 4.4], it follows that

$$\Lambda^1 - k_n = \Lambda - k_n = \text{supp}(\mu_n) \xrightarrow{w} \text{supp}(\mu) = \Gamma = \Gamma^1.$$

(Here it is crucial that the multiplicities  $m_\Lambda(\lambda), m_\Gamma(\gamma)$  are integers.) It remains to show that  $\Lambda^j - k_n \xrightarrow{w} \Gamma^j$  for  $j > 1$ . Since  $\Lambda^1 - k_n \xrightarrow{w} \Gamma^1$ , the case without multiplicity implies that  $\sum_{\lambda \in \Lambda} \delta_{\lambda - k_n} \rightarrow \sum_{\gamma \in \Gamma} \delta_\gamma$ . Therefore,

$$\sum_{\lambda \in \Lambda} (m_\Lambda(\lambda) - 1) \delta_{\lambda - k_n} \rightarrow \sum_{\gamma \in \Gamma} (m_\Gamma(\gamma) - 1) \delta_\gamma.$$

Since  $(\Lambda, m_\Lambda)$  has finite height, we can proceed by induction. Indeed, we consider the sets  $\Lambda_0 := \Lambda^2$  and  $\Gamma_0 := \Gamma^2$ , with multiplicities  $m_{\Lambda_0} := m_\Lambda - 1$  and  $m_{\Gamma_0} := m_\Gamma - 1$ , and note that  $\Lambda_0^j = \Lambda^{j+1}$  and  $\Gamma_0^j = \Gamma^{j+1}$ .  $\square$

## 7.2 Sketch of a Proof of Theorem 2.1

Let  $I := \{(\lambda, j) \in \mathbb{R}^2 : \lambda \in \Lambda^j, j = 1, \dots, N\}$ , and consider the matrix  $A \in \mathbb{C}^{I \times \mathbb{Z}}$ , given by

$$A_{(\lambda, j), k} := G^j(\lambda - k).$$

Then  $\tilde{\Lambda}$  is a sampling set for  $V(G)$  if and only if  $A : \ell^p(\mathbb{Z}) \rightarrow \ell^p(I)$  is bounded below. The independence of  $p$  of this property for the range  $p \in [1, +\infty]$  follows from (a slight extension of) Sjöstrand's Wiener-type lemma [25]. The formulation in [15, Proposition A.1] is applicable directly. Specifically, [15, Proposition A.1] concerns a matrix indexed by two *relatively separated* subsets of the Euclidean space (where a relatively separated set is just a finite union of separated sets). In our case,  $I$  is a relatively separated subset of  $\mathbb{R}^2$ , while  $\mathbb{Z}$  can be embedded into  $\mathbb{R}^2$  as  $\mathbb{Z} \times \{0\}$ . This accounts for the equivalences (a)  $\Leftrightarrow$  (b). The other implications follow, with very minor modifications, as in the proof of [15, Theorem 3.1]. See also [13, Section 4] for some relevant technical tools.  $\square$

## 7.3 Sketch of a Proof of Proposition 3.7

The proposition follows from the theory of density of frames. The Paley–Wiener case is explicitly treated in [14] following the technique of Ramanathan and Steger [20]. For shift-invariant spaces with generators in  $g \in W_0^N(\mathbb{R})$ , we can use the abstract density results for frames from [6] as follows.

Suppose that  $(\Lambda, m_\Lambda)$  is a sampling set for  $V^2(g)$ . By assumption, the Bessel map,  $\ell^2(\mathbb{Z}) \ni c \rightarrow \sum_k c_k g(\cdot - k) \in V^2(g)$ , is an isomorphism. The sampling inequality (1.2) with  $p = 2$  means that the set  $\mathcal{F}$  formed by the sequences

$$\varphi_{\lambda,j} := \left( g^{(j)}(\lambda - k) \right)_{k \in \mathbb{Z}}, \quad \lambda \in \Lambda, j = 0, \dots, m_\Lambda(\lambda) - 1,$$

is a frame for  $\ell^2(\mathbb{Z})$ . We consider the index set  $I := \{(\lambda, j) \in \mathbb{R}^2 : \lambda \in \Lambda, j = 0, \dots, m_\Lambda(\lambda) - 1\}$  and a map  $\alpha : I \rightarrow \mathbb{Z}$  such that  $\alpha(\lambda, j) = l$ , with  $|l - \lambda| \leq 1/2$ . Second, we let  $\Phi(x) := \sum_{j=0}^{N-1} \max_{y: |y-x| \leq 1} |g^{(j)}(y)|$ . Since  $g \in W_0^N(\mathbb{R})$ , it follows that  $\Phi \in W_0(\mathbb{R})$ , and we have the estimate

$$|\varphi_{\lambda,j}(k)| = |g^{(j)}(\lambda - k)| \leq \Phi(\alpha(\lambda) - k),$$

which, in the terminology of [6], means that  $\mathcal{F}$  is  $\ell^1$ -localized with respect to the canonical basis of  $\ell^2(\mathbb{Z})$ . The comparison theorem [6, Thm. 3] yields the estimate  $D^-(I, \alpha) \geq 1$  in terms of the density

$$D^-(I, \alpha) = \liminf_{n \rightarrow \infty} \inf_{k \in \mathbb{Z}} \frac{\#\alpha^{-1}([k-n, k+n])}{\#[k-n, k+n]}.$$

Clearly  $D^-(I, \alpha)$  coincides with  $D^-(\Lambda, m_\Lambda)$ .

Alternative arguments can be given by checking the general conditions in [12] or [22].  $\square$

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