



# Perturbative Yang–Mills theory without Faddeev–Popov ghost fields

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## ABSTRACT

A modified Faddeev–Popov path integral density for the quantization of Yang–Mills theory in the Feynman gauge is discussed, where contributions of the Faddeev–Popov ghost fields are replaced by multi-point gauge field interactions. An explicit calculation to  $O(g^2)$  shows the equivalence of the usual Faddeev–Popov scheme and its modified version.

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## 1. Introduction

Faddeev and Popov [1] proposed a highly acclaimed path integral quantization procedure for Yang–Mills theory. Yang–Mills theory is a gauge theory based on compact simple Lie groups. It forms the basis of our understanding of the Standard Model of particle physics [2–4], which has two basic components: The spontaneously broken  $SU(2) \times U(1)$  electroweak theory, and the unbroken  $SU(3)$  color gauge theory, known as Quantum Chromodynamics (QCD).

Although electromagnetism and the weak interactions (responsible for the forces between sub-atomic particles that cause radioactive decay) appear quite different at everyday low energies, the Standard Model understands them as two different aspects of the same force. The Higgs mechanism [5–9] provides an explanation for the presence of massive gauge bosons (the carriers of the weak force) aside of the massless photon (the carrier of the electromagnetic force). The discoveries of the massive  $W^\pm$  and  $Z$  gauge bosons at the CERN  $\bar{p}p$  collider [10–13] as well as of the Higgs particle at the Large Hadron Collider [14,15] are considered as major successes for the European Organization for Nuclear Research.

QCD is the theory which describes the strong interactions between massive quarks and massless gluons (responsible for binding neutrons and protons to create atomic nuclei). QCD exhibits two main properties, asymptotic freedom [16,17] and color confinement. Asymptotic freedom refers to the weakness of the strong interactions at short-distances (or high energies, respectively). It allows a perturbative treatment, which is often referred to as perturbative QCD. Hereby high energy hadronic processes involving a

large momentum transfer can be factorized into one part which requires detailed nonperturbative information on parton distribution functions and into a second part, which is calculable using perturbation theory. Parton distribution functions (specifying how hadrons are built out of quarks and gluons) have to be extracted from data and are available from various groups worldwide. The perturbative part of the calculation is done in an expansion in the coupling constant. In this context, we especially mention results at next-to-leading order [18], next-to-next-to-leading order [19–22], even next-to-next-to-next-to-leading order [23,24], as well as calculations supplemented with resummations of logarithmic contributions [25,26]. Confinement is the phenomenon of non-observation of color charged particles like free quarks or gluons and is believed to follow from the strength of the QCD coupling constant at long distances (or low energies, respectively). It should be remarked, however, that presently there is no analytic proof of color confinement in Yang–Mills theory. Confinement is crucial for explaining why nuclear forces are short ranged while massless gluon exchange would be long ranged: Nucleons are colorless so they cannot exchange colored gluons but only colorless states. The lightest such particles are pions, which fixes the range of nuclear forces by the inverse of their mass to about  $10^{-14}$  cm.

Upon quantizing Yang–Mills theory new fields are introduced, called Faddeev–Popov ghost fields, which are associated to the gauge fixing. Mathematically, ghost fields allow for an integral representation of the Faddeev–Popov determinant (see below) in terms of a local action functional. The ghost fields are Lorentz scalars but obey Fermi statistics, they are arising inside Feynman diagrams in closed loops only. The ghosts' unphysical degrees of freedom are needed to exactly cancel unphysical polarization states of the gauge field, leading to a unitary theory. The proof of unitarity relies on the Slavnov–Taylor identities [27,28], which in turn play a key role in the proof of the renormalizability [29,30]

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of Yang–Mills theories. The Slavnov–Taylor identities led Becchi, Rouet and Stora [31,32] and, independently, Tyutin [33] to discover a global supersymmetry invariance of the gauge fixed Yang–Mills action including the ghost contributions.

The perturbation theory of Yang–Mills theory as developed by Faddeev–Popov, the property of asymptotic freedom and renormalizability are at the heart of the Standard Model of elementary particle physics.

In this paper a modified Faddeev–Popov path integral quantization of Yang–Mills theory is presented, where contributions of the Faddeev–Popov ghost fields are replaced by multi-point gauge field interactions. This is a new formulation of quantum Yang–Mills theory without the use of Grassman-valued fields.

## 2. Local features of Yang–Mills theory

Let  $\mathcal{A}$  be the space of Yang–Mills fields and  $\mathcal{G}$  the gauge group (for a detailed mathematical account of the involved space we refer to [34–36]). Then  $\mathcal{G}$  defines a principal  $\mathcal{G}$ -bundle  $\mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G} =: \mathcal{M}$  over the space  $\mathcal{M}$  of all inequivalent gauge potentials with projection  $\pi$ .  $\mathcal{M}$  represents the true degrees of freedom. However, the principal  $\mathcal{G}$ -bundle  $\mathcal{A} \rightarrow \mathcal{M}$  is not globally trivializable [34–36], giving rise to the so-called Gribov ambiguity [37] (for a recent review see [38]).

It is advantageous to separate the Yang–Mills fields  $A_\mu$  into gauge independent and gauge dependent degrees of freedom. As this is only locally possible due to the non triviality of the bundle  $\mathcal{A} \rightarrow \mathcal{M}$ , we consider the trivializable bundle  $\pi^{-1}(U) \rightarrow U$ , where  $U$  denotes a sufficiently small neighborhood in  $\mathcal{M}$ .

Under a gauge transformation  $\Omega \in \mathcal{G}$  the Yang–Mills field transforms according to

$$A_\mu^\Omega = \Omega A_\mu \Omega^{-1} - \frac{i}{g} (\partial_\mu \Omega) \Omega^{-1}, \quad (1)$$

where  $g$  denotes the Yang–Mills coupling constant. In terms of the local gauge fixing surface

$$\Gamma = \{B_\mu \in \pi^{-1}(U) \mid \partial_\mu B_\mu = 0\} \quad (2)$$

all gauge fields in  $\pi^{-1}(U)$  have the form  $B_\mu^\Omega$ , where  $B_\mu \in \Gamma$  and  $\Omega \in \mathcal{G}$ . Conversely, given any  $A_\mu \in \pi^{-1}(U)$ , there exists a uniquely defined  $\Omega(A) \in \mathcal{G}$  such that  $A_\mu^{\Omega(A)^{-1}} \in \Gamma$ . This explicitly means that  $\Omega(A)$  has to obey

$$0 = \partial_\mu (\Omega(A)^{-1} A_\mu \Omega(A)) - \frac{i}{g} \partial_\mu ((\partial_\mu \Omega(A)^{-1}) \Omega(A)). \quad (3)$$

This equation may be solved for  $\Omega(A)$  as a formal power series in the gauge field  $A_\mu$  [39–42], which will be used in the next section.

## 3. Perturbative Yang–Mills theory without Faddeev–Popov ghost fields

To begin with, let us recall the usual Faddeev–Popov formula [1] for calculating expectation values of gauge invariant observables  $f$

$$\langle f \rangle = \frac{\int \mathcal{D}A \delta(\partial_\mu A_\mu) \det \mathcal{F}_A e^{-S_{\text{inv}}[A]} f(A)}{\int \mathcal{D}A \delta(\partial_\mu A_\mu) \det \mathcal{F}_A e^{-S_{\text{inv}}[A]}}, \quad (4)$$

displaying integrations over unconstrained gauge fields  $A_\mu \in \pi^{-1}(U)$  and delta functions imposing the gauge fixing condition. Here  $\det \mathcal{F}_A = \det \partial_\mu D_\mu(A)$  denotes the determinant of the Faddeev–Popov operator and  $D_\mu(A)$  is the covariant derivative with respect to  $A_\mu$ . The gauge invariant Yang–Mills action  $S_{\text{inv}}[A]$  reads

$$S_{\text{inv}}[A] = \frac{1}{2} \int d^d x \text{Tr} (F_{\mu\nu} F_{\mu\nu}) \quad (5)$$

and is defined in terms of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \quad (6)$$

Note that in order to arrive at the Faddeev–Popov formula [1] an infinite gauge group volume had to be canceled between the numerator and denominator of the expression (4).

We prefer to represent the Faddeev–Popov formula (4) in the equivalent form [43,44]

$$\langle f \rangle = \frac{\int \mathcal{D}B \det \mathcal{F}_B e^{-S_{\text{inv}}[B]} f(B)}{\int \mathcal{D}B \det \mathcal{F}_B e^{-S_{\text{inv}}[B]}}, \quad (7)$$

where the path integral is performed over constrained gauge fields  $B_\mu \in \Gamma$ , and where  $\det \mathcal{F}_B = \det \partial_\mu D_\mu(B)$  denotes the determinant of the Faddeev–Popov operator with respect to  $B_\mu$ .

Inspired by the stochastic quantization scheme [45–47] a generalization of the Faddeev–Popov formula was proposed in [48], where

$$\langle f \rangle = \frac{\int \mathcal{D}B \det \mathcal{F}_B e^{-S_{\text{inv}}[B]} f(B)}{\int \mathcal{D}B \det \mathcal{F}_B e^{-S_{\text{inv}}[B]}} \frac{\int \mathcal{D}\Omega e^{-S_G[\Omega]} f(\Omega)}{\int \mathcal{D}\Omega e^{-S_G[\Omega]}}. \quad (8)$$

Here  $S_G \in C^\infty(\mathcal{G})$  is an arbitrary functional on  $\mathcal{G}$ , such that  $e^{-S_G}$  is integrable with respect to the invariant measure  $\mathcal{D}\Omega$  on  $\mathcal{G}$ . For different modifications of the Faddeev–Popov formula see [40,49].

When evaluated on gauge invariant observables all additional finite contributions of the gauge degrees of freedom due to  $S_G$  cancel out, therefore the generalized definition (8) of expectation values equals (7), which in turn is equivalent to the usual Faddeev–Popov formula (4).

It is our intention, however, *not* to cancel these finite contributions, but to transform the fields  $B_\mu \in \Gamma$  and  $\Omega \in \mathcal{G}$  back into the original variables  $A_\mu \in \pi^{-1}(U)$ . In this case the Jacobian of the field transformation eliminates the Faddeev–Popov determinant, so we obtain [48]

$$\langle f \rangle = \frac{\int \mathcal{D}A e^{-S_{\text{inv}}[A] - S_G[\Omega(A)]} f(A)}{\int \mathcal{D}A e^{-S_{\text{inv}}[A] - S_G[\Omega(A)]}}. \quad (9)$$

Mind that now the path integral is performed over unconstrained gauge fields  $A_\mu \in \pi^{-1}(U)$ , similarly as in (4). Due to the absence of the Faddeev–Popov determinant, however, Faddeev–Popov ghost fields are not present any longer.

In this work we suggest to specify  $S_G$  as

$$S_G[\Omega(A)] = \frac{1}{g^2} \int d^d x \text{Tr} \left( (\partial_\mu \theta(A)_\mu) (\partial_\nu \theta(A)_\nu)^\dagger \right), \quad (10)$$

where

$$\theta(A)_\mu = (\partial_\mu \Omega(A)^{-1}) \Omega(A) \quad (11)$$

is defined in terms of  $\Omega(A)$ .

To accommodate in  $S_G[\Omega(A)]$  the explicit expression for  $\Omega(A)$  is an involved task and can only be achieved in a perturbative expansion in the coupling constant  $g$ . Although  $S_G[\Omega(A)]$  is depending on the gauge fields  $A_\mu$  in a highly intricate manner, the path integral (9) itself is performed over unconstrained gauge fields  $A_\mu \in \pi^{-1}(U)$ .

It will be seen that our choice for  $S_G$  is implying gauge field propagators in the Feynman gauge. In a sequel paper we plan to study also the covariant  $\xi$ -gauges as well as the limiting  $\xi \rightarrow 0$  case of the Landau gauge, when multiplying  $S_G$  by the inverse of a gauge fixing parameter  $\xi$ .

With the parametrization  $\Omega(A) = e^{i\nu}$  one finds  $\nu = g \nu_1 + g^2 \nu_2 + \dots$  with

$$\partial^2 \nu_1 = \partial_\mu A_\mu \quad (12)$$

and

$$\partial^2 \nu_2 = i\partial_\mu \left( \frac{1}{2} [\nu_1, \partial_\mu \nu_1] - [\nu_1, A_\mu] \right). \quad (13)$$

Correspondingly, one obtains the contributions to the gauge fixing action  $\mathcal{S}_G = \mathcal{S}_G^0 + \mathcal{S}_G^1 + \mathcal{S}_G^2 + \dots$  in a power series expansion of the coupling constant  $g$ .

To lowest order we have

$$\mathcal{S}_G^0 = \int d^d x \text{Tr} \left( (\partial_\mu A_\mu)^2 \right), \quad (14)$$

which, as advocated, represents the standard gauge fixing term of Yang–Mills theory in the Feynman gauge. This standard gauge fixing term, however, is accompanied by additional, unconventional interaction terms of the gauge field. To the first order in  $g$  a new triplic gauge field interaction term arises

$$\mathcal{S}_G^1 = -ig \int d^d x \text{Tr} \left( (\partial_\mu A_\mu) \partial_\nu [\nu_1, A_\nu] \right) + h.c., \quad (15)$$

whereas the second order expansion in  $g$  provides us with new quartic gauge field interaction terms

$$\begin{aligned} \mathcal{S}_G^2 = & -ig^2 \int d^d x \text{Tr} \left( (\partial_\mu A_\mu) \partial_\nu [\nu_2, A_\nu] \right) \\ & - \frac{1}{2} g^2 \int d^d x \text{Tr} \left( (\partial_\mu [\nu_1, A_\mu]) (\partial_\nu [\nu_1, A_\nu]) \right) \\ & - \frac{1}{2} g^2 \int d^d x \text{Tr} \left( (\partial_\mu A_\mu) \partial_\nu [\nu_1, [\nu_1, A_\nu]] \right) + h.c. \end{aligned} \quad (16)$$

$\Omega(A)$ , as well as the gauge fixing action  $\mathcal{S}_G[\Omega(A)]$  defined in (10), can in principle be calculated in perturbation theory to any desired order in  $g$ , implying higher and higher multi-point gauge field interaction terms. Feynman rules corresponding to these interactions may be derived in addition to the standard three-point and four-point Yang–Mills gauge field terms.

When calculating expectation values of gauge invariant observables the new interaction terms will generate the usual Faddeev–Popov ghost contributions order by order in perturbation theory. To explicitly verify this general claim we choose  $F_{\mu\nu} F_{\mu\nu}$  as gauge invariant observable. Contributions to  $\mathcal{O}(g^2)$  arising from (15) and (16) read

$$\langle F^2 \rangle_{\text{new}} = (gf^{abc})^2 \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} A(p, q, r) B(p, q, r), \quad (17)$$

where  $f^{abc}$  are the structure constants and






$$A(p, q, r) = (2\pi)^d \delta^{(d)}(p + q + r) \frac{p^2 q^2 - (p \cdot q)^2}{p^2 q^2 r^2}. \quad (18)$$

The subscript “new” refers to contributions arising only from the new three- and four-point gauge field interaction vertices. The function  $B(p, q, r)$  singles out the different contributions from the various diagrams, see Table 1.

Summing up all terms precisely gives the usual contribution of the Faddeev–Popov ghost fields. We employed Mathematica as supplement to the involved calculations done by hand. Details of the procedure, including the new Feynman rules, will be reported elsewhere.

**Table 1**

Contributions to  $B(p, q, r)$ . The solid dot represents the new three-point gauge field interaction vertex and the solid square the new four-point gauge field interaction vertex. Vertices without any special labeling correspond to the standard Yang–Mills ones. The wiggly gauge field lines represent propagators in the Feynman gauge. The bottom line of the table displays the standard Faddeev–Popov ghost field contribution.

	$-\frac{2}{r^2}$
	$\frac{q^2 + r^2}{p^2 r^2}$
	$\frac{2p^2 - q^2}{p^2 r^2}$
	$-\frac{2}{p^2}$
	$-\frac{1}{p^2}$

#### 4. Outlook

Perturbative Yang–Mills theory without Faddeev–Popov ghost fields has been presented in this paper and shown to be viable and useful. We are confident that various generalizations and a new arena of promising applications will open up.

First we plan doing perturbative calculations for various sets of observables. The efficiency of our method and its general performance in comparison to the conventional Faddeev–Popov procedure will be studied.

We expect to extend our method for calculations in the covariant  $\xi$ -gauge as well as for the limiting  $\xi \rightarrow 0$  case of the Landau gauge.

In another approach we envisage to adapt our scheme for covariant gauge-fixing procedures in lattice gauge theories [50–55], for a review see [56].

A further challenge will be to formulate generalized Slavnov–Taylor identities [27,28] and set up a perturbative renormalization program.

Finally we propose to discuss our method beyond its local perturbative validity, addressing the issue of Gribov ambiguities. Similar to [37,57,58] the functional integration could be restricted to a subset of the gauge field space. Alternatively we suggest the possibility of partitioning the whole space of gauge fields into patches – where gauge fixing without Gribov ambiguity is possible locally – and summing appropriately over all patches [59].

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