Thin Objects: An Abstractionist Account

by Øystein Linnebo

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REVIEWED BY INGER BAKKEN PEDERSEN

Realism within philosophy of mathematics comes in different guises. It concerns both the language of mathematics, so that mathematical propositions have objective truth values, and the subject matter of mathematics, so that propositions purport to describe and quantify over objectively existing objects. The two combined have usually been connected to mathematical platonism and the views of Plato, Gottlob Frege, and Kurt Gödel. On the object realist side of mathematical platonism, one finds a certain analogy between mathematical reality and physical reality, so that the existence of mathematical objects is likened to the existence of physical objects. The existence of the number 2 is thus allied to the existence of the chair I am sitting on. Since the former is abstract and the latter exists in space-time, this relatedness appears unlikely.

In pondering existence, two points spring to mind. The first is whether something exists independently or only in the context of something else. If abstract mathematical objects exist similarly to physical objects, they should exhibit some quality of dependent existence that mirrors the dependent existence of physical objects. Let us look at the example of the chair again, and what we could call the science of chairs. There are many ways in which the chair as object can be said to depend on something else for its existence. One is to say that the chair depends on the atoms of which it is constituted and their specific arrangement. Another is to say that the existence of the chair depends on its chair-like function, i.e., its particular role in the collection called furniture. In both cases, the dependencies are causal. In the former, it is rather obvious. In the latter case, i.e., dependent existence due to the chair's function, we see that this dependent existence warrants the causal label as well; the chair has its chair-like function because we consider chairs within a web made up of physical, historical, and cultural practice. Since all this takes place in space-time, the science of chairs is thoroughly causal, and the existence of the chair on which I sit depends on certain causal facts and relations.

Since mathematical objects are abstract, the possibility of their relying on some causal facts is precluded. What about their so-called role or function within the collection of mathematical objects, say the number 2 within the collection of the natural numbers? We could, in the vein of mathematical structuralism, argue that the number 2 relies on its place in the natural number structure, so that its existence partly or wholly depends on its relations to the other natural numbers and the structure as a whole. But this is also not causal dependency as in the case of the chair and its place within the collection of furniture. To ask whether the existence of a mathematical object is independent of or dependent on something else is rather a question of metaphysical dependence. The relation of grounding expresses this. Grounding is a dependence relation that is not causal, but expresses a relation of metaphysically explaining something, i.e., accounting for something else's existence or properties. If an object is grounded in another, there is usually a question of fundamentality, so that the object in which something is grounded is more fundamental than the object grounded in it. The relation of grounding thus suits the dependence relation mathematical objects might have, since it does not turn on the question of causality. The dependent existence of mathematical objects does not, then, mirror the dependent existence of physical objects, neither for the facts on which they depend nor for the role they have in the respective collections to which they belong.

The second point that springs to mind is how the existence of mathematical objects relates to human beings. We want to shed light on how we view and think about the existence of mathematical objects, so that we can successfully refer to them and have justified beliefs and knowledge about them. These questions constitute the epistemological line of inquiry. Our relationship to abstract objects is marred by the lack of a clear-cut epistemic access to them. If we were to consider mathematical objects as existing similarly to physical objects, it would mean that our epistemic access to mathematical objects should mirror our way of accessing physical objects. Attempts to ensure such access in a mirroring fashion—Gödel's faculty of mathematical intuition being a prime example—have been largely unsuccessful. Furthermore, when Paul Benacerraf presented an apparently fatal objection to mathematical platonism with his epistemological challenge,[1] it seemed inevitable that the coup de grace would be administered shortly, if such had not already been accomplished.

Mathematical realism did not fall on this sword, however, nor does it have to. As Øystein Linnebo puts it in his book *Thin Objects: An Abstractionist Account*, the analogy on which mathematical platonism rests is not, as it were, "entirely appropriate … *as mathematical objects are strikingly different from physical objects*" (p. 190). This we have already seen with the dissimilarity of their dependence relations: physical objects with causal dependence relations and mathematical objects with noncausal dependence relations, but with grounding relations as a possible candidate instead. Object realism in mathematics does not, after all, necessitate endorsing mathematical platonism. The key thus lies in considering mathematical realism as encompassing a larger space, so as not to be forced into a framework in which mathematical and physical reality must

[1] Paul Benacerraf. Mathematical Truth. *Journal of Philosophy* 19 (1973), 661–679.

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be analogous. It is this way forward that brings us to metaontological minimalism and the idea of thin objects.

Øystein Linnebo's *Thin Objects* is an innovative advancement of such efforts. Linnebo begins the book by taking a stand on metaontology. While ontology characterizes what there is, a metaontological line of inquiry targets the core concepts of ontology, such as existence and objecthood (p. 4). How we define certain ontological concepts will largely inform and determine the ontological view in question, since the metaphysical demands on the existence of mathematical objects and what it takes for something to count as an object are integral to any realist position. For Linnebo's project, this metaontological line of inquiry is thus pivotal for the two abovementioned points: metaphysical dependence of objects and possible epistemic access to them. Metaontological minimalism takes a stance on such questions, drastically lowering the bar for existence and objecthood. The ontological view that emerges from such minimalism is the idea of thin objects, which have carried great philosophical promise (p. xi).

What does it mean to consider an object thin? An object can be considered thin in an absolute or relative sense. It is thin in an absolute sense if its existence does not make a substantial demand on the world, and thin in a relative sense if "given the existence of [some other] objects, the existence of the object in question makes no substantial *further* demand" on the world (p. 4). In contrast, spatiotemporal bodies and molecules always are thick; given the existence of the chair I am sitting on, the set of that chair makes no substantial further demand on the world. A mathematical object that is thin in an absolute sense, on the other hand, would be, for instance, the empty set or the natural number 2. In endorsing thin objects, we get two immediate benefits. First, as is attendant to any realist position, there is the issue of the ontological abundance of mathematical objects. In conceiving mathematical objects to be thin, we could alleviate this worry: if the existence of mathematical objects makes no substantial demands on the world, it is easier to explain why mathematics successfully deals with such an unrivaled abundance of objects (compared to any other science) (p. xi). Second, if the bar for existence is lowered, truths about mathematical objects could be more easily attainable (p. xi). Or so the neo-Fregeans Bob Hale and Crispin Wright argue: "The truth of the equinumerosity claim is said to be 'conceptually sufficient' for the truth of the number identity" (p. xi). For instance, in the case of the number of forks and knives on a table (given that they are in one-to-one correspondence), the truth of this equinumerosity claim should suffice for ensuring that a number of forks exists, that a number of knives exists, and that the equality of the two numbers holds.

Linnebo's *Thin Objects* wants to make mathematical reality available to us by extending our knowledge of mathematical objects by steps of abstraction. From old knowledge of supposedly unproblematic facts, we are given sufficient means to "get to" hitherto unknown knowledge. For example, from the simple knowledge that there is a one-to-one correspondence between the forks and knives, we extract the new knowledge of the identity and existence of the numbers of forks and knives. Central to this abstractionist program are both the Fregean triangle of reference, objecthood, and identity criteria and abstraction principles. The Fregean triangle is a picture of how the concepts of objecthood, reference, and identity criteria should be viewed together as interrelated. Objecthood explains the identity criteria, the identity criteria ensure the reference, and the reference suffices for objecthood, so that taken together, they form a triangle of explanatory relations. Linnebo relies here on two core Fregean ideas. The first is that the bar for objecthood is lowered and viewed as the "possible referent of a singular term," and the second is that "singular reference can be … easy to achieve [since] … there is a close link between reference and criteria of identity" (p. xii). This second idea ensures successful reference of a term if it can be linked to a "specification of the would-be referent, which figures in an appropriate criterion of identity" (p. xii). A good way to see this is by considering the following identity criterion for the direction of lines: $d(l_1) = d(l_2) \leftrightarrow l_1 \| l_2$. This is a Fregean example of an abstraction principle, i.e., principles of the form $\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$, where § is an operator that applies to variables like α and β (that are of the same type), and $\sim$ is an equivalence relation on the sort of objects over which the variables range (p. xiii). In the example of the direction of lines, we see that the direction of line 1 is identical to the direction of line 2 if and only if the two lines are parallel. Frege and the neo-Fregeans take this example to mean that the identity of the directions is just another way of saying that the lines are parallel, i.e., it is only another way of recarving the same metaphysical content. From the left-hand side we can get to the right-hand side, and from the right-hand side we can get to the left. The motivation is that such principles can provide a "logical and philosophical foundation for classical mathematics" (p. 8). Linnebo differs from Frege and the neo-Fregeans in that he views such moves of abstraction not to be "recarvings" and furthermore, that the relation need not be a biconditional. He makes use of an asymmetric sufficiency operator $\Rightarrow$, so that the direction example becomes $l_1 \| l_2 \Rightarrow d(l_1) = d(l_2)$ (p. 18). The sufficiency operator now becomes a way to see how we can get new knowledge from old knowledge. From the unproblematic knowledge that two lines are parallel, we are given a new concept, the concept of direction. All that is needed for our new knowledge is a fact about lines; i.e., this suffices for our access to the concept of direction. This constitutes a different metaphysical picture from that of the neo-Fregeans. The sufficiency operator does not go both ways, and so it is an asymmetric picture of abstraction, in which we are given genuinely new objects (the concept of direction) and not simply recarvings of the same content.

An asymmetric picture of abstraction fixes a directed pathway to knowledge of new objects. In order for generalized abstraction principles to provide a logical foundation for mathematics, we need a variant of abstraction that can tackle the main challenge faced by abstractionist programs, namely the so-called bad company problem. The bad company problem designates the difficulty of distinguishing acceptable abstraction principles (e.g., Hume's principle) from unacceptable ones, that is, the "bad companions," e.g., Basic Law V (p. 52). Linnebo aims at pacifying this challenge by way of dynamic abstraction. Dynamic abstraction opposes the neo-Fregeans' static approach, inasmuch as the domain in which the abstraction takes place is single, fixed, and usually taken to contain all of reality (p. 52). With dynamic abstraction, we can expand the domain, so that when "directions are obtained by abstracting on lines … there is no reason to assume that the directions are present already in the domain with which we started" (p. 52). This becomes a way of avoiding paradoxes resulting from "bad" abstraction principles (such as Russell's paradox), since we now allow "*the abstracta to lie outside of the domain on which we abstract*" (p. 52). Linnebo thus favors predicative instead of impredicative abstraction. An abstraction principle is deemed impredicative "if the terms on its left-hand side denote objects included in the range of some quantifier occurring on its right-hand side; otherwise the abstraction principle is *predicative*" (p. 97). A clear example here is again the example of directions, where on the left-hand side, the variables l_1 and l_2 do not range over directions, only lines. The pathway to new knowledge remains fixed from left to right, very much in line with the "worldly asymmetry" of abstraction (p. 18). Furthermore, since the domain expansions can be iterated, what used to be "new" objects can become "old" in the expanded domain. This is why the approach of dynamic abstraction is one of the main advantages of Linnebo's abstractionist program; it relieves us of the bad company problem, exhibits affinities with the iterative conception of sets, and makes "new" objects available while expanding the domain in which we operate.

As already surveyed, Linnebo's book undertakes an abstractionist minimalist variant of logicism, in which the idea of thin objects, the Fregean triangle, and dynamic abstraction take center stage. These three topics constitute the essential first part of the book and also make up the first three chapters. These three chapters are the pillars of Linnebo's ambitious logicist project. This brings us to the second part. Comprising four chapters, it makes comparisons to Frege, Augustin Rayo, and Hale and Wright. It is these chapters that differentiate Linnebo's views from those of Frege—who remains the main inspiration for the book—and other neo-Fregean logicist programs. Chapter 7, on the context principle ("The meaning of a word must be inquired after in propositional context, not in isolation"[2]), is more of a historical paper in Frege scholarship, in which different interpretations of the principle are discussed. It is generally known that the principle is one of three fundamental principles meant to guide Frege's *Grundlagen*, but Linnebo here defends the view that the principle also survives in *Grundgesetze* and should be interpreted as metasemantic (p. 129). While the chapter departs somewhat from the rest of the book in its historically interpretative character, it is very helpful to see the Fregean roots of Linnebo's project fully, and especially how it deals with the ultrathin conception of reference and the idea of recarving of content (pp. 124–125). As we have seen, the recarving of content is central to the asymmetry of Linnebo's abstractionist approach. The third part of the book consists of five chapters, which tackle the details of themes raised earlier in the book, such as "Reference by Abstraction" (Chapter 8), "The Question of Platonism" (Chapter 11), and "Dynamic Set Theory" (Chapter 12). In addition, throughout the book, each chapter has appendices explaining details and stating the proofs mentioned in that chapter. Also, in the preface, Linnebo provides an outline of the interconnections of each chapter and how they can be read efficiently without missing out on the main ideas of the book.

Thin Objects: An Abstractionist Approach continues the revival of logicist and abstractionist efforts and reinvigorates mathematical object realism. In conceiving mathematical objects to be thin, Linnebo establishes a realist view that is less robust, and so avoids the pitfalls attendant to full-fledged platonism. By taking a metaontological minimalist position, using the Fregean triangle to clarify the core concepts of objecthood, reference, and identity criteria, and then to top it off with a technical innovation such as dynamic abstraction, makes for a highly creative and farsighted philosophical undertaking. *Thin Objects* is therefore warmly recommended as a novel contribution to the philosophy of mathematics. It gives a thorough understanding of the abstractionist approach, clearly showing its Fregean roots, but at the same time diverging from Frege and the neo-Fregeans in substantial and innovative ways.

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[2]Gottlob Frege. *Foundations of Arithmetic*, translated by Dale Jacquette. Routledge, 2007, p. 17.

Inger Bakken Pedersen
Department of Philosophy
University of Vienna
Universitätsstraße 7
1010 Vienna
Austria
e-mail: inger.pedersen@univie.ac.at