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Categorical frameworks for generalized functions

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Abstract We tackle the problem of finding a suitable categorical framework for generalized functions used in mathematical physics for linear and non-linear PDEs. We are looking for a Cartesian closed category which contains both Schwartz distributions and Colombeau generalized functions as natural objects. We study Frölicher spaces, diffeological spaces and functionally generated spaces as frameworks for generalized functions. The latter are similar to Frölicher spaces, but starting from locally defined functionals. Functionally generated spaces strictly lie between Frölicher spaces and diffeological spaces, and they form a complete and cocomplete Cartesian closed category. We deeply study functionally generated spaces (and Frölicher spaces) as a framework for Schwartz distributions, and prove that in the category of diffeological spaces, both the special and the full Colombeau algebras are smooth differential algebras, with a smooth embedding of Schwartz distributions and smooth pointwise evaluations of Colombeau generalized functions.

Mathematics Subject Classification 46T30 · 46F25 · 46F30 · 58Dxx

الملخص

نعالج مسألة إيجاد إطار فئوي مناسب لدوال معممة مستخدمة في الفيزياء الرياضية لمعادلات تفاضلية جزئية خطية وغير خطية. نبحث عن فئة مغلقة ديكارتياً تحتوي على توزيعات شوارتز ودوال كولومبو المُعَمَّمَة كأشياء طبيعية. ندرس فضاءات فروليشر وفضاءات ديفيولوجية وفضاءات مولدة داليا كأطر للدوال المعممة. هذه الأخيرة شبيهة بفضاءات فروليشر لكنها مبتدِئة من دالِّيّات معرفة محليا. تقع الفضاءات المولَّدة دالِّيًا بدقَّة بين فضاءات فروليشر والفضاءات الديفيولوجية وتشكل فئة مغلقة ديكارتيّاً تامة وتامة مرافقة. ندرس بتعمق الفضاءات المولدة دالِّياً (وفضاءات فروليشر) كإطار لتوزيعات شوارتز ونبرهن أنه في فئة الفضاءات الديفيولوجية تكون كل من جبريات كولومبو الخاصة والكاملة جبريتين تفاضليتين ملساوين مع تضمين أملس لتوزيعات شوارتز والتقييمات النُّقْطيَّة الملساء لدوال كولومبو المُعَمَّمَة.

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1 Introduction: finding a categorical framework for generalized functions

The problem of considering (generalized) derivatives of locally integrable functions arises frequently in Physics, e.g., in idealized models like in shock mechanics, material points mechanics, charged particles in electrodynamics, gravitational waves in general relativity, etc. (see, e.g., [12,24,34]). Therefore, the need to perform calculations with discontinuous functions like one deals with smooth functions motivated the introduction of generalized functions (GF) as objects extending, in some sense, the notion of function. As such, generalized functions find deep applications in solutions of singular differential equations [1,25,33,35] and are naturally framed in (several) theories of infinite-dimensional spaces, from locally convex vector spaces [27] and convenient vector spaces [29] up to diffeological [26,28] and Frölicher spaces [13].

The foundation of a rigorous linear theory of generalized functions has been pioneered by Schwartz with a deep use of locally convex vector space theory [25,36], and heuristic multiplications of distributions early appeared, e.g., in quantum electrodynamics, elasticity, elastoplasticity, acoustics and other fields [12,34]. Despite the impossibility of a straightforward extension of Schwartz linear theory [37] to an algebra extending pointwise product of continuous functions, the theory of Colombeau algebras (see, e.g., [10–12,24,33,34]) permits to bypass this impossibility in a very simple way by considering an algebra of generalized functions which extends the pointwise product of smooth functions.

The main aim of the present work is to study different categories as frameworks for generalized functions. In particular, we introduce the category **FDlg** of *functionally generated spaces*. This category has very nice properties and strictly lies between the category of Frölicher spaces and the category of diffeological spaces.

We start by recalling the algebras from Colombeau theory that we will consider in this work. Henceforth, we will use the notations of [24,25] for the well-known Schwartz distribution theory.

1.1 The special and full Colombeau algebras

The special Colombeau algebra

In this section, we fix some basic notations and terminology from Colombeau theory. For details we refer to [24]. We include zero in the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Henceforth, Ω will always be an open subset of $\mathbb{R}^n$ and we denote by I the interval $(0, 1]$. The (special) Colombeau algebra on Ω is defined as the quotient $\mathcal{G}^s(\Omega) := \mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega)$ of *moderate nets* over *negligible nets*, where the former is

$$\mathcal{E}^s_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall\alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and the latter is

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall\alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m})\}.$$

Throughout this paper, every asymptotic relation is for $\varepsilon \to 0^+$. Nets in $\mathcal{E}^s_M(\Omega)$ are written as (u_ε), and we use $u = [u_\varepsilon]$ to denote the corresponding equivalence class in $\mathcal{G}^s(\Omega)$. For $(u_\varepsilon) \in \mathcal{N}^s(\Omega)$, we also write $(u_\varepsilon) \sim 0$. Then, $\Omega \mapsto \mathcal{G}^s(\Omega)$ is a fine and supple sheaf of differential algebras, and there exist sheaf embeddings of the space of Schwartz distributions $\mathcal{D}'$ into $\mathcal{G}^s$ (cf. [24]). A very simple way to embed $\mathcal{D}'$ into $\mathcal{G}^s$ is given by the following result [39]:

Theorem 1.1 *There exists a net $(\psi_\varepsilon) \in \mathcal{D}(\mathbb{R}^n)^I$ with the properties:*

(i) $\forall \varepsilon \in I \, \forall x \in supp(\psi_\varepsilon) : \; |x| < 1$;
(ii) $\int \psi_\varepsilon = 1 \, \forall \varepsilon \in I$, *where the implicit integration is over the whole* $\mathbb{R}^n$;
(iii) $\forall \alpha \in \mathbb{N}^n \, \exists N \in \mathbb{N} : \; \sup_{x \in \mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(\varepsilon^{-N})$;
(iv) $\forall j \in \mathbb{N} \, \exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] \, \forall \alpha \in \mathbb{N}^n : 1 \le |\alpha| \le j \Rightarrow \int x^\alpha \psi_\varepsilon(x) \, dx = 0$;
(v) $\forall \eta \in \mathbb{R}_{>0} \, \exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \; \int |\psi_\varepsilon| \le 1 + \eta$.

In particular, if we set

$$\varepsilon \odot \psi_\varepsilon : x \in \mathbb{R}^n \mapsto \frac{1}{\varepsilon^n} \psi_\varepsilon \left(\frac{x}{\varepsilon} \right) \in \mathbb{R} \quad \forall \varepsilon \in I$$

$$\iota_\Omega(u) := [u * (\varepsilon \odot \psi_\varepsilon|_\Omega)] \quad \forall u \in \mathcal{D}'(\Omega) \tag{1.1}$$

then we have:

(vi) $\iota_\Omega : \mathcal{D}'(\Omega) \longrightarrow \mathcal{G}^s(\Omega)$ *is a linear embedding;*
(vii) $\partial^\alpha(\iota_\Omega(u)) = \iota_\Omega(D^\alpha u)$ *for all* $u \in \mathcal{D}'(\Omega)$ *and all* $\alpha \in \mathbb{N}^n$, *where* D^α *and* ∂^α *are the* α*-partial differential operators on* $\mathcal{D}'(\Omega)$ *and* $\mathcal{G}^s(\Omega)$, *respectively;*
(viii) $\iota_\Omega(f) = [f]$ *for all* $f \in \mathcal{C}^\infty(\Omega)$.

The ring of constants in $\mathcal{G}^s$ is denoted by $\widetilde{\mathbb{R}}$ or $\widetilde{\mathbb{C}}$, and is called the *ring of Colombeau generalized numbers* (CGN). It is an ordered ring with respect to the order defined by $[x_\varepsilon] \le [y_\varepsilon]$ iff $\exists [z_\varepsilon] \in \widetilde{\mathbb{R}}$ such that $(z_\varepsilon) \sim 0$ and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for ε sufficiently small. This order is not total, but we can still define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, and analogously the supremum of two elements. More generally, the space of generalized points in Ω is $\widetilde{\Omega} = \Omega_M / \sim$, where $\Omega_M = \{(x_\varepsilon) \in \Omega^I \mid \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N})\}$ is called the *set of moderate nets*, and $(x_\varepsilon) \sim (y_\varepsilon)$ if $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^m)$ for every $m \in \mathbb{N}$. By $\mathcal{N}$ we denote the set of all negligible nets of real numbers $(x_\varepsilon) \in \mathbb{R}^I$, i.e., such that $(x_\varepsilon) \sim 0$.

The space of compactly supported generalized points $\widetilde{\Omega}_c$ is defined by $\Omega_c / \sim$, where $\Omega_c := \{(x_\varepsilon) \in \Omega^I \mid \exists K \Subset \Omega \, \exists \varepsilon_0 \, \forall \varepsilon < \varepsilon_0 : \; x_\varepsilon \in K\}$, and $\sim$ is the same equivalence relation as in the case of $\widetilde{\Omega}$. Any Colombeau generalized function (CGF) $u \in \mathcal{G}^s(\Omega)$ acts on the generalized points from $\widetilde{\Omega}_c$ by $u(x) := [u_\varepsilon(x_\varepsilon)]$ and is uniquely determined by its point values (in $\widetilde{\mathbb{R}}$) on the compactly supported generalized points [24], but not on the standard points. A CGF $[u_\varepsilon]$ is called *compactly bounded* (c-bounded) from Ω into Ω' if for any $K \Subset \Omega$, there exists $K' \Subset \Omega'$ such that $u_\varepsilon(K) \subseteq K'$ for ε small. This type of CGF is closed with respect to compositions. Moreover, if $u \in \mathcal{G}^s(\Omega)$ is c-bounded from Ω into Ω' and $v \in \mathcal{G}^s(\Omega')$, then $[v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}^s(\Omega)$. For $x, y \in \widetilde{\mathbb{R}}^n$, we write $x \approx y$ if $x - y$ is infinitesimal, i.e., if $|x - y| \le r$ for all $r \in \mathbb{R}_{>0}$.

Topological methods in Colombeau theory are usually based on the so-called *sharp topology* (see e.g., [3] and references therein), which is the topology generated by the balls $B^{\mathrm{s}}_\rho(x) = \{y \in \widetilde{\mathbb{R}}^n \mid |y - x| < \rho\}$, where $| - |$ is the natural extension of the Euclidean norm on $\widetilde{\mathbb{R}}^n$, i.e., $|[x_\varepsilon]| := [|x_\varepsilon|] \in \widetilde{\mathbb{R}}$, and $\rho \in \widetilde{\mathbb{R}}_{>0}$ is positive invertible. Henceforth, we will also use the notation $\widetilde{\mathbb{R}}^* := \{x \in \widetilde{\mathbb{R}} \mid x \text{ is invertible}\}$. Finally, Garetto in [14,15] extended the above construction to arbitrary locally convex spaces by functorially assigning a space $\mathcal{G}^s_E$ of CGF to any given locally convex space E. The seminorms of E can then be used to define pseudovaluations which in turn induce a generalized locally convex topology on the $\widetilde{\mathbb{C}}$-module $\mathcal{G}^s_E$, again called the *sharp topology*.

The full Colombeau algebra

Clearly, the embedding ι_Ω defined in (1.1) depends on the net of maps $(\psi_\varepsilon) \in \mathcal{D}(\mathbb{R}^n)^I$ whose existence is given by Theorem 1.1. This shall not be considered only in a negative way: e.g., it is not difficult to choose (ψ_ε) so that the embedding satisfies the properties that $H(0) = \iota_\mathbb{R}(H)(0) = [\int_{-\infty}^0 \psi_\varepsilon] = 0$ and $\delta(0) = \iota_\mathbb{R}(\delta)(0) = [\psi_\varepsilon(0)]$ is an infinite number of $\widetilde{\mathbb{R}}$ (here, H is the Heaviside function and δ is the Dirac delta function). These properties are informally used in several applications.

The main idea of the full Colombeau algebra is to consider a different set of indices, instead of $I = (0, 1]$, so as to obtain an intrinsic embedding.

Definition 1.2 (i) $\mathcal{A}_0(\Omega) := \{\phi \in \mathcal{D}(\Omega) \mid \int \phi = 1\}$ and $\mathcal{A}_0 := \mathcal{A}_0(\mathbb{R}^n)$;
(ii) $\mathcal{A}_q(\Omega) := \{\phi \in \mathcal{A}_0(\Omega) \mid \forall \alpha \in \mathbb{N}^n : \; 1 \le |\alpha| \le q \Rightarrow \int x^\alpha \phi(x) \, dx = 0\}$ and $\mathcal{A}_q := \mathcal{A}_q(\mathbb{R}^n)$;

(iii) $U(\Omega) := \{(\phi, x) \in \mathcal{A}_0 \times \Omega \mid \mathrm{supp}(\phi) \subseteq \Omega - x\}$;
(iv) We say that $R \in \mathcal{E}^e(\Omega)$ iff $R : U(\Omega) \longrightarrow \mathbb{R}$ and

$$\forall \phi \in \mathcal{A}_0 : \ R(\phi, -) \text{ is smooth on } \Omega \cap \{x \in \mathbb{R}^n \mid \mathrm{supp}(\phi) \subseteq \Omega - x\};$$

(v) We say that $R \in \mathcal{E}^e_M(\Omega)$ iff $R \in \mathcal{E}^e(\Omega)$ and

$$\forall K \Subset \Omega \, \forall \alpha \in \mathbb{N}^n \, \exists N \in \mathbb{N} \, \forall \phi \in \mathcal{A}_N : \ \sup_{x \in K} |\partial^\alpha R(\varepsilon \odot \phi, x)| = O(\varepsilon^{-N});$$

(vi) We say that $R \in \mathcal{N}^e(\Omega)$ iff $R \in \mathcal{E}^e(\Omega)$ and

$$\forall K \Subset \Omega \, \forall \alpha \in \mathbb{N}^n \, \forall m \in \mathbb{N} \, \exists q \in \mathbb{N} \, \forall \phi \in \mathcal{A}_q : \ \sup_{x \in K} |\partial^\alpha R(\varepsilon \odot \phi, x)| = O(\varepsilon^{m});$$

(vii) $\mathcal{G}^e(\Omega) := \mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega)$ is called the *full Colombeau algebra*;
(viii) The above-mentioned intrinsic embedding $\iota_\Omega : \mathcal{D}'(\Omega) \longrightarrow \mathcal{G}^e(\Omega)$ is defined by $(\iota_\Omega u)(\phi, x) := \langle u, \phi(\cdot - x)\rangle$. It verifies properties like (vi), (vii) and (viii) in Theorem 1.1.

For motivations and details, see [24].

2 Functionally generated diffeologies

2.1 Preliminaries on diffeological spaces and Frölicher spaces

Both diffeological spaces and Frölicher spaces are generalizations of smooth manifolds, introduced by J.M. Souriau and A. Frölicher, respectively, in the 1980s. The smooth structure (called the *diffeology*) on a diffeological space is defined by some testing functions from all open subsets of all Euclidean spaces to the given set, subject to a covering condition, a presheaf condition and a sheaf condition (see Definition 2.1). A possible intuitive description of this structure on a diffeological space X is that a diffeology is the specification not only of a particular family of smooth functions (like charts on manifolds), but of all the possible smooth maps of the type $d : U \longrightarrow X$ for all open subsets $U \subseteq \mathbb{R}^n$ and for all $n \in \mathbb{N}$. We can roughly say that we have to specify what are smooth curves, surfaces, etc., on the space X.

On a Frölicher space X, we consider only $U = \mathbb{R}$, i.e., the smooth structure on the space is given by a set of smooth curves; moreover, these curves are determined by (and they determine) a given set of functionals, i.e., of smooth functions of the type $l : X \longrightarrow \mathbb{R}$ (see Definition 2.7). The category **Fr** of all Frölicher spaces is a full subcategory of the category **Dlg** of all diffeological spaces.

In the following subsections, we are going to focus on a family of diffeological spaces called *functionally generated (diffeological) spaces*, where the diffeological structure is determined by a given family of locally defined smooth functionals. As we will see in the present work that, these spaces frequently appear in functional analysis, strictly lie between diffeological spaces and Frölicher spaces, and the category **FDlg** of all these spaces behaves nicely—it is complete, cocomplete and Cartesian closed.

To simplify the notation, we write $\mathcal{O}\mathbb{R}^\infty$ for the category of open sets in Euclidean spaces and ordinary smooth functions.

Definition 2.1 A *diffeological space* $X = (|X|, \mathcal{D})$ is a set $|X|$ together with a specified family of functions

$$\mathcal{D} = \cup_{U \in \mathcal{O}\mathbb{R}^\infty} \mathcal{D}_U \quad \text{with } \mathcal{D}_U \subseteq \mathbf{Set}(U, |X|)$$

such that for any $U, V \in \mathcal{O}\mathbb{R}^\infty$, the following three axioms hold:

(i) Every constant function $d : U \longrightarrow |X|$ is in $\mathcal{D}_U$ (Covering condition);
(ii) $d \circ f \in \mathcal{D}_V$ for any $d : U \longrightarrow |X| \in \mathcal{D}_U$ and any $f \in \mathcal{C}^\infty(V, U)$ (Presheaf condition);
(iii) Let $d \in \mathbf{Set}(U, |X|)$, and let $\{U_i\}_{i \in I}$ be an open covering of U. Then, $d \in \mathcal{D}_U$ if $d|_{U_i} \in \mathcal{D}_{U_i}$ for each $i \in I$ (Sheaf condition).

For a diffeological space $X = (|X|, \mathcal{D})$, every element in $\mathcal{D}$ is called a *plot* of X. We write $d \in_U X$ to denote that $d \in \mathcal{D}_U$, which will also be called a *figure* of type U of the space X.

Definition 2.2 A morphism (also called a *smooth map*) $f : X \longrightarrow Y$ between two diffeological spaces $X = (|X|, \mathcal{D}^X)$ and $Y = (|Y|, \mathcal{D}^Y)$ is a function $|f| : |X| \longrightarrow |Y|$ such that $f \circ d \in \mathcal{D}^Y_U$ for any $d \in \mathcal{D}^X_U$ and $U \in \mathcal{O}\mathbb{R}^\infty$.

If we write $f(d) := f \circ d$, by the covering condition of Definition 2.1, we have a generalization of the usual evaluation; moreover, $f : X \longrightarrow Y$ is smooth if and only if for all $U \in \mathcal{O}\mathbb{R}^\infty$ and $d \in_U X$, we have $f(d) \in_U Y$, i.e., f take figures of type U on the domain to figures of the same type in the codomain. Moreover, $X = Y$ as diffeological spaces if and only if for all d and U, $d \in_U X$ if and only if $d \in_U Y$. These and several other generalizations of set-theoretical properties justify the use of the symbol $\in_U$.

All diffeological spaces with smooth maps form a category, which will be denoted by **Dlg**. Given two diffeological spaces X and Y, we write $\mathcal{C}^\infty(X, Y)$ for the set of all smooth maps $X \longrightarrow Y$.

Here is a list of basic properties of diffeological spaces. We refer readers to the standard textbook [26] for more details.

Remark 2.3 (i) By a smooth manifold, we always assume that it is Hausdorff, finite-dimensional and without boundary. Every smooth manifold M is automatically a diffeological space $\mathbf{M} = (M, \mathcal{D})$ with $d \in_U \mathbf{M}$ if and only if $d : U \longrightarrow M$ is smooth in the usual sense. We call this $\mathcal{D}$ the *standard diffeology* on M, and without specification, we always assume that a smooth manifold is equipped with this diffeology when viewed as a diffeological space. Moreover, given two smooth manifolds M and N, $f : \mathbf{M} \longrightarrow \mathbf{N}$ is smooth if and only if $f : M \longrightarrow N$ is smooth in the usual sense. In other words, the category **Man** of all smooth manifolds and smooth maps is fully embedded in **Dlg**. This justifies our notation $\mathcal{C}^\infty(X, Y)$ for the hom-set $\mathbf{Dlg}(X, Y)$. Limits of smooth manifolds that already exist in **Man** are preserved by this embedding (see Theorem 2.25). Generally speaking the same property does not hold for colimits of smooth manifolds that already exist in **Man**.

(ii) Given a set X, the set of all diffeologies on X forms a complete lattice. The smallest diffeology is called the *discrete diffeology*, which consists of all locally constant functions, and the largest diffeology is called the *indiscrete diffeology*, which consists of all set functions. Let $A = (X, \mathcal{D}_A)$ and $B = (X, \mathcal{D}_B)$ be two diffeological spaces with the same underlying set. We simply write $A \subseteq B$ iff $1_X : A \longrightarrow B$ is smooth, i.e., iff $\mathcal{D}_A \subseteq \mathcal{D}_B$.
Therefore, given a family of functions $\mathfrak{I} := \{\iota_i : |X_i| \longrightarrow Y\}_{i \in I}$ from the underlying sets of the diffeological spaces X_i to a fixed set Y, there exists a smallest diffeology on Y making all these maps ι_i smooth. We call this diffeology, the *final diffeology associated to* $\mathfrak{I}$. In more detail,

$$d \in_U Y \text{ iff } \forall u \in U\, \exists V \text{ neigh. of } u\, \exists i \in I\, \exists \delta \in_V X_i :\ \iota_i \circ \delta = d|_V.$$

Dually, given a family of functions $\mathfrak{J} := \{p_j : X \longrightarrow |Y_j|\}_{j \in J}$ from a given set X to the underlying sets of the diffeological spaces Y_j, there exists a largest diffeology on X making all these maps p_j smooth. We call this diffeology the *initial diffeology associated to* $\mathfrak{J}$. In more detail,

$$d \in_U X \text{ iff } p_j \circ d \in_U Y_j\ \forall j \in J.$$

In particular, if Y is a quotient set of $|X|$, then the final diffeology on Y associated to the quotient map $|X| \longrightarrow Y$ is called the *quotient diffeology*, and Y with the quotient diffeology is called a *quotient diffeological space* of X. Dually, if X is a subset of $|Y|$, then the initial diffeology on X associated to the inclusion map $X \longrightarrow |Y|$ is called the *sub-diffeology*, and we write $(X \prec Y)$ to denote this new diffeological space. We call $(X \prec Y)$ the *diffeological subspace* of Y. Finally, the initial diffeology associated to the projection maps $p_i : \prod_{i \in I} |X_i| \longrightarrow |X_i|$ of an arbitrary product is called the *product diffeology*, and dually the final diffeology associated to the inclusion maps $|X_j| \longrightarrow \coprod_{j \in J} |X_j|$ of an arbitrary coproduct is called the *coproduct diffeology*.

(iii) The category **Dlg** is complete and cocomplete. In more detail, let $G : \mathcal{I} \longrightarrow \mathbf{Dlg}$ be a functor from a small category $\mathcal{I}$. Write $|-| : \mathbf{Dlg} \longrightarrow \mathbf{Set}$ for the forgetful functor. Then, both $\lim G$ and $\operatorname{colim} G$ exist in **Dlg** as lifting and co-lifting of limits and colimits in **Set**. In more detail, $|\lim G| = \lim |G|$ and the diffeology of $\lim G$ is the initial diffeology associated to the universal cone $\{\lim |G| \longrightarrow |G(i)|\}_{i \in \mathcal{I}}$ in **Set**; dually $|\operatorname{colim} G| = \operatorname{colim} |G|$ and the diffeology of $\operatorname{colim} G$ is the final diffeology associated to the universal co-cone $\{|G(i)| \longrightarrow \operatorname{colim} |G|\}_{i \in \mathcal{I}}$ in **Set**.

(vi) The category **Dlg** is Cartesian closed. In more detail, given three diffeological spaces X, Y and Z, there is a canonical diffeology (called the *functional diffeology*) on $\mathcal{C}^\infty(X, Y)$ defined by

$$d \in_U \mathcal{C}^\infty(X, Y) \quad \text{iff } d^\vee \in \mathcal{C}^\infty(U \times X, Y),$$

with $d^\vee(u, x) = d(u)(x)$ (in the present work, we use the notations of [2]). Without specification, the set $\mathcal{C}^\infty(X, Y)$ is always equipped with the functional diffeology when viewed as a diffeological space. Then, Cartesian closedness means that $f \in \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z))$ if and only if $f^\vee \in \mathcal{C}^\infty(X \times Y, Z)$ (or, equivalently that $g \in \mathcal{C}^\infty(X \times Y, Z)$ if and only if $g^\wedge \in \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z))$, where $g^\wedge(x)(y) := g(x, y)$). Therefore, Cartesian closedness permits to equivalently translate an infinite-dimensional problem like $f \in \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z))$ into a finite-dimensional one like $f^\vee \in \mathcal{C}^\infty(X \times Y, Z)$, and vice versa.

(v) Every diffeological space can be extended with infinitely near points $X \in \mathbf{Dlg} \mapsto {}^\bullet X \in {}^\bullet\mathbf{Dlg}$, $X \subseteq {}^\bullet X$, obtaining a non-Archimedean framework similar to Synthetic Differential Geometry (see e.g., [31] and references therein), but compatible with the classical logic. The category ${}^\bullet\mathbf{Dlg}$ of *Fermat spaces* is defined by generalizing the category of diffeological spaces, but taking suitable smooth functions defined on the extension ${}^\bullet U \subseteq {}^\bullet\mathbb{R}^n$ of open sets $U \in \mathcal{O}\mathbb{R}^\infty$. It is remarkable to note that the so-called *Fermat functor* ${}^\bullet(-) : \mathbf{Dlg} \longrightarrow {}^\bullet\mathbf{Dlg}$ has very good preservation properties strictly related to the intuitionistic logic. See [16–20,23] for more details.

(vi) **Dlg** is a quasi-topos, and hence is locally Cartesian closed [4].

Every diffeological space has an interesting canonical topology:

Definition 2.4 Let $X = (|X|, \mathcal{D})$ be a diffeological space. The final topology τ_X induced by $\mathcal{D}$ is called the *D-topology*.

Without specification, every diffeological space X is equipped with the D-topology τ_X. Elements in τ_X are called *D-open subsets*.

Example 2.5 (i) The D-topology on any smooth manifold is the usual topology.
(ii) The D-topology on any discrete (indiscrete) diffeological space is the discrete (indiscrete) topology.

Theorem 2.6 [40] $T_\mathrm{D} : \mathbf{Dlg} \longrightarrow \mathbf{Top}$ *defined by* $T_\mathrm{D}(X) = (|X|, \tau_X)$ *is a functor,*[1] *which has a right adjoint* $\mathrm{D}_\mathrm{T} : \mathbf{Top} \longrightarrow \mathbf{Dlg}$ *defined by* $|\mathrm{D}_\mathrm{T}(X)| := |X|$ *and* $d \in_U \mathrm{D}_\mathrm{T}(X)$ *iff* $d \in \mathbf{Top}(U, X)$ *(both functors act as identity on arrows).*

As a consequence, the D-topology of a quotient diffeological space of X is same as the quotient topology of $T_\mathrm{D}(X)$. However, the D-topology of a diffeological subspace of X may be different from the sub-topology of $T_\mathrm{D}(X)$.

For more detailed discussion about the D-topology of diffeological spaces, see [26, Chapter 2] and [8].

Now, let us turn to Frölicher spaces. In several spaces of functional analysis (like all those listed in Sect. 2.5), smooth figures are "generated by smooth functionals". Therefore, smoothness can also be tested using smooth functionals, similarly to using projections in finite-dimensional Euclidean spaces. In Frölicher spaces, we focus our attention also to smooth functions of the type $X \longrightarrow \mathbb{R}$.

Definition 2.7 A Frölicher space $(\mathcal{C}, X, \mathcal{F})$ is a set X together with two specified families of functions

$$\mathcal{C} \subseteq \mathbf{Set}(\mathbb{R}, X) \quad \text{and} \quad \mathcal{F} \subseteq \mathbf{Set}(X, \mathbb{R})$$

with the following smooth compatibility conditions:

$$c : \mathbb{R} \longrightarrow X \in \mathcal{C} \quad \text{iff } l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\ \forall l \in \mathcal{F},$$

and

$$l : X \longrightarrow \mathbb{R} \in \mathcal{F} \quad \text{iff } l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\ \forall c \in \mathcal{C}.$$

Definition 2.8 A morphism $f : (\mathcal{C}_X, X, \mathcal{F}_X) \longrightarrow (\mathcal{C}_Y, Y, \mathcal{F}_Y)$ between two Frölicher spaces is a function $f : X \longrightarrow Y$ such that one of the following equivalent conditions holds:

(i) $f \circ c \in \mathcal{C}_Y\ \forall c \in \mathcal{C}_X$;
(ii) $l \circ f \in \mathcal{F}_X\ \forall l \in \mathcal{F}_Y$;
(iii) $l \circ f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\ \forall c \in \mathcal{C}_X$ and $\forall l \in \mathcal{F}_Y$.

[1] We can recall the symbol T_D by saying "topological space from diffeological space". Analogously, we can recall the plenty of symbols for the other functors related to the categories in this paper.

All Frölicher spaces and their morphisms form a category, which will be denoted by **Fr**. Here is a list of basic properties for Frölicher spaces. For details, we refer readers to [13,30].

Remark 2.9 (i) Every smooth manifold M is automatically a Frölicher space with $\mathcal{C} = \mathcal{C}^\infty(\mathbb{R}, M)$ and $\mathcal{F} = \mathcal{C}^\infty(M, \mathbb{R})$. Without specification, we always assume that a smooth manifold is equipped with this Frölicher structure when viewed as a Frölicher space. Moreover, this gives a full embedding of **Man** into **Fr**.

(ii) Let $\mathfrak{I} := \{\iota_i : X_i \longrightarrow Y\}_{i\in I}$ be a family of functions from the underlying sets of the Frölicher spaces $(\mathcal{C}_i, X_i, \mathcal{F}_i)$ to a fixed set Y. Let

$$\mathcal{F}_Y = \{l : Y \longrightarrow \mathbb{R} \mid l \circ \iota_i \in \mathcal{F}_i \ \forall i\},$$

and let

$$\mathcal{C}_Y = \{c : \mathbb{R} \longrightarrow Y \mid l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \ \forall l \in \mathcal{F}_Y\}.$$

Then, $(\mathcal{C}_Y, Y, \mathcal{F}_Y)$ is a Frölicher space and all these maps ι_i are morphisms between Frölicher spaces. We call this Frölicher structure on Y the *final Frölicher structure associated to* $\mathfrak{I}$.
Dually, let $\mathfrak{J} := \{p_j : X \longrightarrow Y_j\}_{j\in J}$ be a family of functions from a fixed set X to the underlying sets of the Frölicher spaces $(\mathcal{C}_j, Y_j, \mathcal{F}_j)$. Let

$$\mathcal{C}_X = \{c : \mathbb{R} \longrightarrow X \mid p_j \circ c \in \mathcal{C}_j \ \forall j\},$$

and let

$$\mathcal{F}_X = \{l : X \longrightarrow \mathbb{R} \mid l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \ \forall c \in \mathcal{C}_X\}.$$

Then, $(\mathcal{C}_X, X, \mathcal{F}_X)$ is a Frölicher space and all these maps p_j are morphisms between Frölicher spaces. We call this Frölicher structure on X the *initial Frölicher structure associated to* $\mathfrak{J}$.

(iii) The category **Fr** is complete and cocomplete. In more detail, let $G : \mathcal{I} \longrightarrow$ **Fr** be a functor from a small category $\mathcal{I}$. Write $|-| :$ **Fr** $\longrightarrow$ **Set** for the forgetful functor. Then, both $\lim G$ and $\operatorname{colim} G$ exist in **Fr** as lifting and co-lifting of limits and colimits in **Set**. In more detail, $|\lim G| = \lim |G|$ and the Frölicher structure of $\lim G$ is the initial Frölicher structure associated to the universal cone $\{\lim |G| \longrightarrow |G(i)|\}_{i\in\mathcal{I}}$ in **Set**; dually $|\operatorname{colim} G| = \operatorname{colim} |G|$ and the Frölicher structure of $\operatorname{colim} G$ is the final Frölicher structure associated to the universal co-cone $\{|G(i)| \longrightarrow \operatorname{colim} |G|\}_{i\in\mathcal{I}}$. In the category **HFr** of Hausdorff Frölicher spaces, limits and colimits of smooth manifolds that already exist in **Man** are preserved by the embedding **Man** $\longrightarrow$ **HFr** (see Theorem 2.29).

(iv) The category **Fr** is Cartesian closed. In more detail, given Frölicher spaces X and Y, set $\mathcal{C} = \{c : \mathbb{R} \longrightarrow \mathbf{Fr}(X, Y) \mid c^\vee \in \mathbf{Fr}(\mathbb{R} \times X, Y)\}$, and $\mathcal{F} = \{l : \mathbf{Fr}(X, Y) \longrightarrow \mathbb{R} \mid l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \ \forall c \in \mathcal{C}\}$. Then, one can show that $(\mathcal{C}, \mathbf{Fr}(X, Y), \mathcal{F})$ is a Frölicher space. Without specification, $\mathbf{Fr}(X, Y)$ is always equipped with this Frölicher structure when viewed as a Frölicher space.

(v) Given a Frölicher space $(\mathcal{C}, Y, \mathcal{F})$, let

$$\mathcal{D}_U = \{d : U \longrightarrow Y \mid l \circ d \in \mathcal{C}^\infty(U, \mathbb{R}) \ \forall l \in \mathcal{F}\}.$$

Then, $\mathrm{D_F}(\mathcal{C}, Y, \mathcal{F}) := (Y, \mathcal{D} = \cup_{U\in\mathcal{O}\mathbb{R}^\infty} \mathcal{D}_U)$ is a diffeological space. This defines a full embedding $\mathrm{D_F}$: **Fr** $\longrightarrow$ **Dlg**. So, there will be no confusion to call *smooth maps* also the morphisms between Frölicher spaces. Moreover, one shows that $\mathrm{D_F}(\mathbf{Fr}(A, B)) = \mathcal{C}^\infty(\mathrm{D_F}(A), \mathrm{D_F}(B))$ as diffeological spaces. This embedding functor has a left adjoint given as follows. For a diffeological space $X = (|X|, \mathcal{D})$, let $\mathcal{F} = \mathcal{C}^\infty(X, \mathbb{R})$ and let

$$\mathcal{C} = \{c : \mathbb{R} \longrightarrow X \mid l \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \ \forall l \in \mathcal{F}\}.$$

Then, $\mathrm{F_D}(X) := (\mathcal{C}, |X|, \mathcal{F})$ is a Frölicher space. Both functors $\mathrm{D_F}$ and $\mathrm{F_D}$ are identities on the morphisms. For more discussion on the relationship between diffeological spaces and Frölicher spaces, see [5,38].

2.2 Definition and examples of functionally generated diffeologies

Now, let us introduce a special class of diffeological spaces called *functionally generated spaces*, which are like Frölicher spaces, but with locally defined smooth functionals. The idea is that, in this type of spaces, we can determine whether a continuous function $d : U \longrightarrow T_{\mathrm{D}}(X)$ is a figure of X by testing the smoothness of its composition with a given family of smooth functions $l : (A \prec X) \longrightarrow \mathbb{R}$.

Definition 2.10 Let $X = (|X|, \mathcal{D})$ be a diffeological space, and let $\mathcal{F} = \{\mathcal{F}_A\}_{A\in\tau_X}$ be a τ_X-family of smooth functions, i.e., for each $A \in \tau_X$

$$\mathcal{F}_A \subseteq \mathcal{C}^\infty((A \prec X), \mathbb{R}).$$

We say that $\mathcal{F}$ *generates* $\mathcal{D}$ if for any open set $U \in \mathcal{O}\mathbb{R}^\infty$ and any continuous map $d : U \longrightarrow T_{\mathrm{D}}(X)$, the condition

$$\forall A \in \tau_X\, \forall l \in \mathcal{F}_A :\ l \circ d|_{d^{-1}(A)} \in \mathcal{C}^\infty(d^{-1}(A), \mathbb{R}) \tag{2.1}$$

implies that $d \in_U X$, i.e., that d is a plot of X. Any map $l \in \mathcal{C}^\infty((A \prec X), \mathbb{R})$ is called a *locally defined smooth functional* of the space X. Finally, we say that the diffeological space X is *functionally generated* if its diffeology can be generated by some family $\mathcal{F}$, and we denote by **FDlg** the full subcategory of **Dlg** of all functionally generated (diffeological) spaces.

If the codomain of a continuous map $f : X \longrightarrow Y$ is functionally generated, then we can also test the smoothness of f by locally defined smooth functionals of Y:

Theorem 2.11 *Let $f : |X| \longrightarrow |Y|$ be a map with $X \in$ **Dlg** and $Y \in$ **FDlg**. Assume that the diffeology of Y is generated by the family $\{\mathcal{F}_A\}_{A\in\tau_Y}$. Then, the following are equivalent*

(i) $f \in \mathcal{C}^\infty(X, Y)$;
(ii) $f \in \mathbf{Top}(T_{\mathrm{D}}(X), T_{\mathrm{D}}(Y))$ *and*

$$\forall A \in \tau_Y\, \forall l \in \mathcal{F}_A :\ l \circ f|_{f^{-1}(A)} \in \mathcal{C}^\infty((f^{-1}(A) \prec X), \mathbb{R}). \tag{2.2}$$

Proof Since the implication (i) ⇒ (ii) is clear, we only prove the opposite one. For any $d \in_U X$, since $d \in \mathbf{Top}(U, T_{\mathrm{D}}(X))$, $f \circ d \in \mathbf{Top}(U, T_{\mathrm{D}}(Y))$. Then for any $A \in \tau_Y$ and any $l \in \mathcal{F}_A$, by (2.2), we get $l \circ f|_{f^{-1}(A)} \in \mathcal{C}^\infty((f^{-1}(A) \prec X), \mathbb{R})$, and hence

$$l \circ f|_{f^{-1}(A)} \circ d|_{d^{-1}(f^{-1}(A))} = l \circ (f \circ d)|_{(f\circ d)^{-1}(A)} \in \mathcal{C}^\infty((f \circ d)^{-1}(A), \mathbb{R}).$$

Since the diffeology of Y is generated by $\{\mathcal{F}_A\}_{A\in\tau_Y}$, the conclusion $f \circ d \in_U Y$ follows. □

Here is a list of basic properties and examples of functionally generated spaces:

Remark 2.12 (i) The notion of functionally generated space is of local nature, i.e., we can equivalently say that $\mathcal{F}$ generates $\mathcal{D}$ if for any $U \in \mathcal{O}\mathbb{R}^\infty$ and any $d \in \mathbf{Set}(U, |X|)$, the condition

$$\begin{aligned} &\forall u \in U\, \forall A \in \tau_X\, \forall l \in \mathcal{F}_A :\ d(u) \in A \\ &\quad \Rightarrow \exists V \text{ neigh. of } u :\ d(V) \subseteq A\ ,\ l \circ d|_V \in \mathcal{C}^\infty(V, \mathbb{R}) \end{aligned}$$

implies $d \in_U X$.

(ii) We can also equivalently require that $\mathcal{F}_A \subseteq \mathbf{Set}(A, \mathbb{R})$, and for all continuous maps $d : U \longrightarrow T_{\mathrm{D}}(X)$ for all $U \in \mathcal{O}\mathbb{R}^\infty$, we have $d \in_U X$ if and only if (2.1) holds. Therefore, locally defined smooth functionals of a functionally generated space determine completely the figures (plots) of the underlying diffeological space, i.e., if $\mathcal{D}_1$, $\mathcal{D}_2$ are diffeologies on $|X|$ and $\mathcal{F}$ generates both $\mathcal{D}_1$ and $\mathcal{D}_2$, then $\mathcal{D}_1 = \mathcal{D}_2$.

(iii) Let $\mathcal{F}$ generate $\mathcal{D}$. Define $\mathcal{M}^X_A := \mathcal{C}^\infty((A \prec X), \mathbb{R})$ for any $A \in \tau_X$. Then, $\mathcal{M}^X$ also generates $\mathcal{D}$. Of course, $\mathcal{M}^X$ is the maximum family of locally defined smooth functionals of X which can be used to test whether a continuous map $d : U \longrightarrow T_{\mathrm{D}}(X)$ is a figure or not, and the interesting problem is to find a smaller family $\mathcal{F} \subseteq \mathcal{M}^X$ generating the same set of plots of X.

(iv) The diffeology generated by a Frölicher space $(\mathcal{C}, X, \mathcal{F})$ is functionally generated by globally defined smooth functionals. That is, it suffices to consider $\tilde{\mathcal{F}}$ defined by $l \in \tilde{\mathcal{F}}_A$ if and only if $A = X$ and $l \in \mathcal{F}$. Therefore, the functor $\mathrm{F_D} : \mathbf{Fr} \longrightarrow \mathbf{Dlg}$ has values in **FDlg**. In particular, every smooth manifold and every discrete diffeological space is functionally generated. However, there are functionally generated spaces which do not come from Frölicher spaces; see Example 2.23.

In a functionally generated space, besides the usual D-topology τ_X, we can consider the initial topology $\tau_{\mathcal{F}}$ with respect to all locally defined smooth functionals $\bigcup_{A\in\tau_X}\mathcal{F}_A$ of X (which is analogous to the weak topology; see e.g., [27]). In particular, the topology $\tau_{\mathcal{M}^X}$ is called the *functional topology* on X. In general, $\tau_{\mathcal{F}}$ is coarser than the D-topology (see Example 2.23), but in every functionally generated space the functional topology and the D-topology coincide, as stated in the following theorem:

Theorem 2.13 *Let $\mathcal{F}$ generate the diffeology of a space $X \in$ **Dlg**. Then, $\tau_{\mathcal{F}} \subseteq \tau_X$. If $\mathcal{F}_A \neq \emptyset$ for every $A \in \tau_X$, then $\tau_{\mathcal{F}} = \tau_X$. In particular, $\tau_{\mathcal{M}^X} = \tau_X$.*

Proof The topology $\tau_{\mathcal{F}}$ has the set

$$\{l^{-1}(V) \mid l \in \mathcal{F}_A, V \in \tau_{\mathbb{R}}, A \in \tau_X\}$$

as a subbasis. For any $l^{-1}(V)$ in this subbasis and any plot $d \in_U X$, the set

$$d^{-1}(l^{-1}(V)) = (l \circ d|_{d^{-1}(A)})^{-1}(V)$$

is open in U since $l \circ d|_{d^{-1}(A)}$ is smooth by Definition 2.10. Hence, $\tau_{\mathcal{F}} \subseteq \tau_X$. On the other hand, if $A \in \tau_X$ and $l \in \mathcal{F}_A \neq \emptyset$ is any locally defined smooth functional, then $A = l^{-1}(\mathbb{R}) \in \tau_{\mathcal{F}}$. So $\tau_X \subseteq \tau_{\mathcal{F}}$ if $\mathcal{F}_A \neq \emptyset$ for every $A \in \tau_X$. □

Here are some examples of diffeological spaces which are not functionally generated.

Example 2.14 (i) Let $(X, \mathcal{D})$ be the irrational torus $\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$, with the quotient diffeology $\mathcal{D}$; see [26]. Then, $(X, \mathcal{D})$ is not functionally generated. Indeed, since $\mathbb{Z}+\theta\mathbb{Z}$ is dense in $\mathbb{R}$, the D-topology τ_X is indiscrete, i.e., $\tau_X = \{\emptyset, X\}$. Hence, every smooth map $X \longrightarrow \mathbb{R}$ is constant. Therefore, for any function $d : U \longrightarrow X$ and for any $l \in \mathcal{M}^X$, the composition $l \circ d$ is constant, hence smooth. Therefore, there does not exist a family $\mathcal{F}$ that generates $\mathcal{D}$.

(ii) For any $n \geq 2$, let $\mathbb{R}^n_{\mathrm{w}} = (\mathbb{R}^n, \mathcal{D}^{\mathrm{w}})$ be $\mathbb{R}^n$ with the wire diffeology $\mathcal{D}^{\mathrm{w}}$; see [26]. Then, the D-topology of $\mathbb{R}^n_{\mathrm{w}}$ is the usual Euclidean topology, and by Boman's theorem [6], $\mathcal{C}^\infty((A \prec \mathbb{R}^n_{\mathrm{w}}), \mathbb{R}) = \mathcal{C}^\infty((A \prec \mathbb{R}^n), \mathbb{R}) = \mathcal{M}_A^{\mathbb{R}^n_{\mathrm{w}}}$. Hence, if $\mathbb{R}^n_{\mathrm{w}}$ is functionally generated, then we would have $1_{\mathbb{R}^n} \in_{\mathbb{R}^n} \mathbb{R}^n_{\mathrm{w}}$, which is false for the wire diffeology. Therefore, $\mathbb{R}^n_{\mathrm{w}}$ is not functionally generated.

2.3 Categorical properties of functionally generated spaces

In this subsection, we are going to prove some nice categorical properties for the category **FDlg** of functionally generated spaces and smooth maps. That is, **FDlg** is complete, cocomplete and Cartesian closed.

Although the family $\mathcal{F}$ that generates a diffeology is a τ_X-family of smooth functions, in practice, we usually only need a $\mathcal{B}$-family with $\mathcal{B} \subseteq \tau_X$. In other words, $\mathcal{F}_A$ can be any subset (in particular, the empty set) of $\mathcal{C}^\infty((A \prec X), \mathbb{R})$ if $A \in \tau_X \setminus \mathcal{B}$. We have already met such examples in (iv) of Remark 2.12. Here is another big class of examples:

Theorem 2.15 *Let $\{p_i : X \longrightarrow |X_i|\}_{i\in I}$ be a family of functions from a given set X to the underlying sets of the diffeological spaces $X_i = (|X_i|, \mathcal{D}^i)$. Assume that each $\mathcal{D}^i$ is generated by $\mathcal{F}^i$, and let $\mathcal{D}$ be the initial diffeology on X associated to this family (i.e., $d \in_U \mathcal{D}$ iff $p_i \circ d \in_U \mathcal{D}^i$ $\forall i$). For all $A \in \tau_{(X,\mathcal{D})}$, set $l \in \mathcal{F}_A$ iff $l \in \mathcal{C}^\infty((A \prec X), \mathbb{R})$ and*

$$\exists i \in I\ \exists B \in \tau_{X_i}\ \exists \lambda \in \mathcal{F}^i_B : \ A = p_i^{-1}(B)\ ,\ l = \lambda \circ p_i|_A.$$

Then, the diffeology $\mathcal{D}$ is generated by $\mathcal{F}$.

The proof follows directly from Definition 2.10. Note that $\mathcal{F}_A = \emptyset$ if A is not of the form $A = p_i^{-1}(B)$ for some $i \in I$ and $B \in \tau_{X_i}$. So we are essentially considering only smooth functionals defined on the D-open subsets in $\mathcal{B} = \{p_i^{-1}(B) \mid B \in \tau_{X_i}, i \in I\} \subseteq \tau_{(X,\mathcal{D})}$. In this sense, $\mathcal{F}$ is also the smallest family of locally defined smooth functionals generating $(X, \mathcal{D})$ and containing all smooth functions of the form $\lambda \circ p_i|_{p_i^{-1}(B)}$.

In particular, every subset of a functionally generated space with the sub-diffeology is again functionally generated. Analogously, every product of functionally generated spaces with the product diffeology is functionally generated. So we have

Corollary 2.16 *The category* **FDlg** *is complete.*

Similarly, one can show that every coproduct of functionally generated spaces with the coproduct diffeology is again functionally generated.

Let $f, g : X \longrightarrow Y$ be smooth maps between functionally generated spaces. In general, the coequalizer in **Dlg** may not be functionally generated. For example, let X be the set $\mathbb{R}$ equipped with the discrete diffeology, and let Y be the set $\mathbb{R}$ equipped with the standard diffeology. Let θ be some irrational number. Fix a representative in $\mathbb{R}$ for each element in the quotient group $\mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$, i.e., define a function $\rho : \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z}) \longrightarrow \mathbb{R}$ such that $\rho(c) \in c$ for all $c \in \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$. Let $f : X \longrightarrow Y$ be the identity function and let $g : X \longrightarrow Y$ be the function defined by $g(r) := \rho(c)$ for all $r \in c \in \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$, i.e., g sends every point in the subset $r+\mathbb{Z}+\theta\mathbb{Z} = c = \rho(c)+\mathbb{Z}+\theta\mathbb{Z}$ to the fixed representative $\rho(c) \in \mathbb{R}$. It is clear that both f and g are smooth because X has only the locally constant figures, and the coequalizer in **Dlg** is the irrational torus because the equivalence relation of $\mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$ is the smallest one where $f(r) = r$ is equivalent to $g(r) = \rho(c)$ for $r \in c$. We already know from (i) of Example 2.14 that the irrational torus is not functionally generated. However, we will show below that the category **FDlg** is cocomplete, and the coequalizer of the above diagram in **FDlg** is the underlying set of the irrational torus with the indiscrete diffeology.

Now, we show how to define a functionally generated diffeology starting from a diffeological space X and a τ_X-family of smooth functions.

Definition 2.17 Let $X = (|X|, \sigma)$ be a topological space, and let $\mathcal{F} = \{\mathcal{F}_B\}_{B\in\mathcal{B}}$ with $\mathcal{B} \subseteq \sigma$ be a $\mathcal{B}$-family of functionals, i.e., $\mathcal{F}_B \subseteq \mathbf{Set}(B, \mathbb{R})$. For $U \in \mathcal{O}\mathbb{R}^\infty$, write $d \in \mathcal{DF}_U$ (or $\mathcal{D}^X\mathcal{F}_U$ if we need to show the dependence of X) if and only if $d \in \mathbf{Top}(U, X)$ and

$$\forall B \in \mathcal{B}\,\forall l \in \mathcal{F}_B : \; l \circ d|_{d^{-1}(B)} \in \mathcal{C}^\infty(d^{-1}(B), \mathbb{R}).$$

We set $\mathcal{DF} := \cup_{U\in\mathcal{O}\mathbb{R}^\infty}\mathcal{DF}_U$ and call $\hat{X}_\mathcal{F} := (|X|, \mathcal{DF})$ the *diffeological space generated by X and $\mathcal{F}$*; see (ii) of Remark 2.18. When Y is a diffeological space, we always apply the above construction with respect to the D-topology (i.e., $X = T_\mathrm{D}(Y)$) and the locally defined smooth functionals (i.e., $\mathcal{F}_B \subseteq \mathcal{C}^\infty((B \prec X), \mathbb{R})$ for all $B \in \mathcal{B}$).

One can show directly from the definitions that

Remark 2.18 In the hypotheses of Definition 2.17, the following properties hold:

(i) We can trivially extend the $\mathcal{B}$-family $\mathcal{F}$ to the whole σ-family by setting $\mathcal{F}_A := \emptyset$ if $A \notin \mathcal{B}$. We will always assume to have extended $\mathcal{F}$ in this way;
(ii) $\mathcal{DF}$ is a diffeology on $|X|$;
(iii) For all $A \in \sigma$ and $l \in \mathcal{F}_A$, we have $l \in \mathcal{C}^\infty((A \prec \hat{X}_\mathcal{F}), \mathbb{R})$;
(iv) The diffeology $\mathcal{DF}$ of $\hat{X}_\mathcal{F}$ is functionally generated by $\mathcal{F}$.

Moreover, if $X = (|X|, \mathcal{D})$ is a diffeological space, then

(v) $\mathcal{D}_U \subseteq \mathcal{DF}_U$, and hence

$$\mathcal{C}^\infty((Y \prec X), \mathbb{R}) \supseteq \mathcal{C}^\infty((Y \prec \hat{X}_{\mathcal{M}^X}), \mathbb{R})\,\forall Y \subseteq |X|.$$

Together with the above Property (iii), we have

$$\mathcal{C}^\infty((A \prec X), \mathbb{R}) = \mathcal{C}^\infty((A \prec \hat{X}_{\mathcal{M}^X}), \mathbb{R})\,\forall A \in \tau_X;$$

(vi) $\mathcal{F}$ generates $\mathcal{D}$ if and only if $\mathcal{D}_U = \mathcal{DF}_U$;
(vii) the D-topology on $\hat{X}_\mathcal{F}$ coincides with the D-topology τ_X on X;
(viii) If $\mathcal{F}$ generates $\mathcal{D}$, then $X = \hat{X}_\mathcal{F}$. So $\widehat{\hat{X}_\mathcal{F}}_\mathcal{F} = \hat{X}_\mathcal{F}$ for any τ_X-family $\mathcal{F}$. And if X is functionally generated, then $X = \hat{X}_{\mathcal{M}^X}$;
(ix) $\mathcal{DM}^X$ is the smallest functionally generated diffeology on $|X|$ containing $\mathcal{D}$.

In particular, if we take $\mathcal{F}$ to be the empty τ_X-family, i.e., $\mathcal{F}_A = \emptyset$ for all $A \in \tau_X$, then $\mathcal{DF}_U = \mathbf{Top}(U, T_D(X))$.

Theorem 2.19 *The inclusion functor* $\mathrm{D_{FG}} : \mathbf{FDlg} \hookrightarrow \mathbf{Dlg}$ *is a right adjoint of the functor* $\mathrm{FG_D} : X \in \mathbf{Dlg} \mapsto \hat{X}_{\mathcal{M}^X} \in \mathbf{FDlg}$ *(both functors act as identity on arrows). Therefore, for all* $X \in \mathbf{Dlg}$ *and* $Y \in \mathbf{FDlg}$*, we have*

$$\mathcal{C}^\infty(X, Y) = \mathcal{C}^\infty(\hat{X}_{\mathcal{M}^X}, Y).$$

We call $\mathrm{FG_D}(X) = (|X|, \mathcal{D}^X\mathcal{M}^X)$ the *functional extension* of X.

Proof It follows by applying Definitions 2.10, 2.17 and Remark 2.18. □

Corollary 2.20 *Let* $G : \mathcal{I} \longrightarrow \mathbf{FDlg}$ *be a functor from a small category* $\mathcal{I}$*. Then,*

$$\mathrm{FG_D}(\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{D_{FG}}(G_i)) \cong \operatorname*{colim}_{i \in \mathcal{I}} G_i.$$

Therefore, the category **FDlg** *is cocomplete.*

Proof Since $\mathrm{FG_D}$ is a left adjoint, it preserves colimits

$$\mathrm{FG_D}(\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{D_{FG}}(G_i)) \cong \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{FG_D}\,(\mathrm{D_{FG}}(G_i)) = \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{FG_D}(G_i).$$

Since $G_i \in \mathbf{FDlg}$, $\mathrm{FG_D}(G_i) = G_i$ by (viii) of Remark 2.18.

Corollary 2.21 *Let* X *be a diffeological space, and let* S *be a D-open subset of* X*. Then,* $\mathrm{FG_D}(S \prec X) = (S \prec \mathrm{FG_D}(X))$.

Proof By [8, Lem. 3.17], $\tau_{(S \prec X)} = \{A \cap S \mid A \in \tau_X\}$ since S is D-open. By (vii) of Remark 2.18 we have $\tau_{\mathrm{FG_D}(X)} = \tau_X$ and hence $T_\mathrm{D}(\mathrm{FG_D}(S \prec X)) = T_\mathrm{D}(S \prec X) = T_\mathrm{D}(S \prec \mathrm{FG_D}(X))$. The smoothness of the identity set map $\mathrm{FG_D}(S \prec X) \longrightarrow (S \prec \mathrm{FG_D}(X))$ follows from Theorem 2.19, and the smoothness of the inverse set map essentially follows from (v) of Remark 2.18. □

Since the coequalizer in **FDlg** in general is different from the coequalizer in **Dlg**, the forgetful functor $\mathrm{D_{FG}} : \mathbf{FDlg} \hookrightarrow \mathbf{Dlg}$ has no right adjoint. Here is another interesting example that a colimit in **FDlg** is different from the corresponding colimit in **Dlg**:

Example 2.22 Let X be the pushout of

$$\mathbb{R} \xleftarrow{\;0\;} \mathbb{R}^0 \xrightarrow{\;0\;} \mathbb{R} \tag{2.3}$$

in **Dlg**. Then, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^0 & \xrightarrow{\;0\;} & \mathbb{R} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{R} & \xrightarrow[\;j\;]{} & \mathbb{R}^2 \end{array}$$

in **Dlg** with $i(x) = (x, 0)$ and $j(y) = (0, y)$. This induces a smooth injective map $X \longrightarrow \mathbb{R}^2$. Write $Y \in \mathbf{Dlg}$ for the image of this map with the sub-diffeology of $\mathbb{R}^2$. One can show that

(i) the induced smooth map $X \longrightarrow Y$ is not a diffeomorphism;
(ii) the D-topology on X (or Y) coincides with the sub-topology of $\mathbb{R}^2$;
(iii) for any open subset A of $\mathbb{R}^2$, $C^\infty((A \cap X) \prec X, \mathbb{R}) = C^\infty((A \cap Y) \prec Y, \mathbb{R})$, which implies that X is not functionally generated;
(iv) Y is Frölicher because **Fr** is closed with respect to subobjects, so $Y \in \mathbf{FDlg}$.

See [9,41] for more details. Hence, by Corollary 2.20, the pushout of (2.3) in **FDlg** is $\mathrm{FG_D}(X) \cong Y \ncong X$.

Now, we show that the embedding $\mathrm{FG_F} : \mathbf{Fr} \longrightarrow \mathbf{FDlg}$ is not essentially surjective. That is, there are functionally generated spaces which are not from Frölicher spaces:

Example 2.23 Let $Y = (-\infty, 0) \cup (0, \infty)$, and let X be the pushout of

$$\mathbb{R} \hookleftarrow Y \hookrightarrow \mathbb{R}$$

in **Dlg**. Then, $C^\infty(X, \mathbb{R}) \cong C^\infty(\mathbb{R}, \mathbb{R})$. Since no element in $C^\infty(X, \mathbb{R})$ can detect the double points at origin, there is no Frölicher space such that its image under the embedding $\mathrm{D_F}$: **Fr** $\longrightarrow$ **Dlg** is X. But since the two structural maps $\mathbb{R} \longrightarrow X$ are injective and open, X is functionally generated. In other words, for any $U \in \mathcal{O}\mathbb{R}^\infty$, $C^\infty(X, \mathbb{R})$ *cannot* detect whether an arbitrary function $U \longrightarrow X$ is smooth, but it *can* detect whether a continuous function $U \longrightarrow X$ is smooth. Moreover, the initial topology on X with respect to $C^\infty(X, \mathbb{R})$ is strictly coarser than the D-topology.

Theorem 2.24 *The category* **FDlg** *is Cartesian closed.*

Proof Since **Dlg** is Cartesian closed and products in **FDlg** are the same as products in **Dlg**, it suffices to show that if X is a diffeological space and Y is a functionally generated space, then the functional diffeology of the space $\mathcal{C}^\infty(X, Y)$ is functionally generated.

We split the proof of the claim into three steps.

Step 1: We prove that if $\mathcal{C}^\infty(\mathbb{R}^n, Y)$ is functionally generated for all $n \in \mathbb{N}$, then $\mathcal{C}^\infty(X, Y)$ is functionally generated.

To prove that $\mathcal{C}^\infty(X, Y)$ is functionally generated, by (vi) of Remark 2.18, for any $d \in_U \mathrm{FG_D}(\mathcal{C}^\infty(X, Y))$ we need to show that $d \in_U \mathcal{C}^\infty(X, Y)$, i.e., that $d^\vee : U \times X \longrightarrow Y$ is smooth. This is equivalent to show that for any plot $p : \mathbb{R}^n \longrightarrow X$, the composition

$$U \times \mathbb{R}^n \xrightarrow{1_U \times p} U \times X \xrightarrow{d^\vee} Y$$

is a plot of Y. This is again equivalent to show that the composition

$$U \xrightarrow{d} \mathcal{C}^\infty(X, Y) \xrightarrow{p^*} \mathcal{C}^\infty(\mathbb{R}^n, Y)$$

is smooth. By assumption $\mathcal{C}^\infty(\mathbb{R}^n, Y)$ is functionally generated, and the map p^* is smooth, so the adjunction $\mathrm{FG_D} \dashv \mathrm{D_{FG}}$ (Theorem 2.19) implies that $p^* : \mathrm{FG_D}(\mathcal{C}^\infty(X, Y)) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, Y)$ is smooth. But $d \in_U \mathrm{FG_D}(\mathcal{C}^\infty(X, Y))$. So $p^* \circ d : U \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, Y)$ is smooth, which proves our first claim.

Step 2: We prove below that if $d : U \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, Y)$ is a continuous map, then the induced function $d^\vee : U \times \mathbb{R}^n \longrightarrow Y$ is continuous.

Let A be a D-open subset of Y, and let $(u, x) \in (d^\vee)^{-1}(A)$. Since $d(u) \in \mathcal{C}^\infty(\mathbb{R}^n, Y)$, $(d(u))^{-1}(A)$ is an open neighborhood of $x \in \mathbb{R}^n$. Take a precompact open neighborhood V of $x \in \mathbb{R}^n$ such that its closure $\bar{V} \subseteq (d(u))^{-1}(A)$. Write $\tilde{A} = \{f \in \mathcal{C}^\infty(\mathbb{R}^n, Y) \mid f(\bar{V}) \subseteq A\}$. Since the D-topology on $\mathcal{C}^\infty(\mathbb{R}^n, Y)$ contains the compact-open topology [8, Prop. 4.2], $\tilde{A}$ is D-open in $\mathcal{C}^\infty(\mathbb{R}^n, Y)$. Hence, $W := d^{-1}(\tilde{A})$ is an open neighborhood of $u \in U$. Therefore, $W \times V$ is an open neighborhood of $(u, x) \in (d^\vee)^{-1}(A)$, which implies that the map $d^\vee$ is continuous.

Step 3: We prove below that $\mathcal{C}^\infty(\mathbb{R}^n, Y)$ is functionally generated.

Let $d \in_U \mathrm{FG_D}(\mathcal{C}^\infty(\mathbb{R}^n, Y))$. We need to show that the induced function $d^\vee : U \times \mathbb{R}^n \longrightarrow Y$ is smooth. From Step 2, we know that $d^\vee$ is continuous. Since Y is functionally generated, it is enough to show that for any D-open subset A of Y and any $l \in \mathcal{C}^\infty((A \prec Y), \mathbb{R})$, the composition

$$(d^\vee)^{-1}(A) \xrightarrow{d^\vee|_{(d^\vee)^{-1}(A)}} A \xrightarrow{l} \mathbb{R}$$

is smooth. For any $(u, x) \in (d^\vee)^{-1}(A)$, we will use the notations $\tilde{A}$, V and W introduced in Step 2. Since smoothness is a local condition, it is enough to show that the composition

$$W \times V \hookrightarrow (d^\vee)^{-1}(A) \xrightarrow{d^\vee|_{(d^\vee)^{-1}(A)}} A \xrightarrow{l} \mathbb{R}$$

is smooth. Equivalently, we need to show that the composition

$$W \xrightarrow{d|_W} (\tilde{A} \prec \mathcal{C}^\infty(\mathbb{R}^n, Y)) \xrightarrow{\text{Res}} \mathcal{C}^\infty(V, A) \xrightarrow{l_*} \mathcal{C}^\infty(V, \mathbb{R})$$

is smooth, where Res is the restriction map. It is easy to see that the map Res : $(\tilde{A} \prec \mathcal{C}^\infty(\mathbb{R}^n, Y)) \longrightarrow \mathcal{C}^\infty(V, A)$ is smooth, so $l_* \circ \text{Res} : (\tilde{A} \prec \mathcal{C}^\infty(\mathbb{R}^n, Y)) \longrightarrow \mathcal{C}^\infty(V, \mathbb{R})$ is smooth. Since $d \in_U \text{FG}_\text{D}(\mathcal{C}^\infty(\mathbb{R}^n, Y))$, we get $d|_W \in_W (\tilde{A} \prec \text{FG}_\text{D}(\mathcal{C}^\infty(\mathbb{R}^n, Y)))$. But both V and $\mathbb{R}$ are Frölicher spaces, so $\mathcal{C}^\infty(V, \mathbb{R})$ is functionally generated, and the adjunction $\text{FG}_\text{D} \dashv \text{D}_\text{FG}$ (Theorem 2.19) implies that the map $l_* \circ \text{Res} : \text{FG}_\text{D}(\tilde{A} \prec \mathcal{C}^\infty(\mathbb{R}^n, Y)) \longrightarrow \mathcal{C}^\infty(V, \mathbb{R})$ is smooth. By Corollary 2.21 we have $\text{FG}_\text{D}(\tilde{A} \prec \mathcal{C}^\infty(\mathbb{R}^n, Y)) = (\tilde{A} \prec \text{FG}_\text{D}(\mathcal{C}^\infty(\mathbb{R}^n, Y)))$, so the conclusion follows. □

2.4 Preservation of limits and (suitable) colimits of smooth manifolds

In this subsection, we are going to discuss the question that if a limit (or colimit) exists in **Man**, the category of smooth manifolds and smooth maps, then is it the same as the corresponding limit (or colimit) in **FDlg**? The statements of the main results and the idea of proofs mainly come from [32].

Theorem 2.25 [32] *Let* $F : \mathcal{I} \longrightarrow$ **Man** *be a functor from a small category. Assume that* $\lim F$ *exists in* **Man**. *Write* FG_M : **Man** $\longrightarrow$ **FDlg** *for the embedding functor. Then,* $\text{FG}_\text{M}(\lim F) \cong \lim(\text{FG}_\text{M} \circ F)$.

Proof By the universal property of limit in **FDlg**, there is a canonical smooth map $\eta : \text{FG}_\text{M}(\lim F) \longrightarrow \lim(\text{FG}_\text{M} \circ F)$.

First, we prove that $|\eta|$ is surjective. Note that any $x \in |\lim(\text{FG}_\text{M} \circ F)|$ corresponds to a smooth map $x : \mathbb{R}^0 \longrightarrow \lim(\text{FG}_\text{M} \circ F)$. So we have a cone $x \longrightarrow F$. Since $\mathbb{R}^0$ is a smooth manifold and $\lim F$ exists in **Man**, by the universal property of limit in **Man** and in **FDlg**, there exists $y : \mathbb{R}^0 \longrightarrow \lim F$ such that $x = \eta \circ \text{FG}_\text{M}(y)$, which implies that $|\eta|$ is surjective.

Next, we prove that $|\eta|$ is injective. If $a, a' \in |\text{FG}_\text{M}(\lim F)|$ such that $|\eta|(a) = |\eta|(a')$, then the two cones $a \longrightarrow F$ and $a' \longrightarrow F$ have the same image in the target. By the universal property of limit in **Man**, $a = a'$.

Finally, we prove that η^{-1} is smooth. Let $d \in_U \lim(\text{FG}_\text{M} \circ F)$. Since the functor FG_M is fully faithful, we get a cone $U \longrightarrow F$. Note that $\text{FG}_\text{M}(U) = U$. By the universal property of limit in **Man** and in **FDlg**, we get a smooth map $f : U \longrightarrow \lim F$ such that $\eta^{-1} \circ d = \text{FG}_\text{M}(f)$. Hence, η^{-1} is smooth.

Therefore, $\text{FG}_\text{M}(\lim F) \cong \lim(\text{FG}_\text{M} \circ F)$. □

Remark 2.26 Note that the category $\mathcal{O}\mathbb{R}^\infty$ with the usual open coverings is a site. Baez and Hoffnung [4] showed that $\mathcal{O}\mathbb{R}^\infty$ is a concrete site, and the category **Dlg** is equivalent to the category **CSh**($\mathcal{O}\mathbb{R}^\infty$) of concrete sheaves over $\mathcal{O}\mathbb{R}^\infty$.

We write **CPre**($\mathcal{O}\mathbb{R}^\infty$), **Pre**($\mathcal{O}\mathbb{R}^\infty$) and **Sh**($\mathcal{O}\mathbb{R}^\infty$) for the category of concrete presheaves over $\mathcal{O}\mathbb{R}^\infty$, the category of presheaves over $\mathcal{O}\mathbb{R}^\infty$ and the category of sheaves over $\mathcal{O}\mathbb{R}^\infty$, respectively. There are embedding functors

$$\begin{array}{ccccccccc} \mathbf{Man} & \longrightarrow & \mathbf{Fr} & \longrightarrow & \mathbf{FDlg} & \longrightarrow & \mathbf{Dlg} & \longrightarrow & \mathbf{CPre}(\mathcal{O}\mathbb{R}^\infty) \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathbf{Sh}(\mathcal{O}\mathbb{R}^\infty) & \longrightarrow & \mathbf{Pre}(\mathcal{O}\mathbb{R}^\infty). \end{array}$$

Moreover, if a limit exists in **Man**, then the corresponding limits in all the other categories listed above are isomorphic to that limit in the corresponding categories.

In general, if a colimit in **Man** exists, when viewed as a functionally generated space it may be different from the corresponding colimit in **FDlg**.

Example 2.27 In Example 2.23, we showed that the underlying set of the pushout X of the diagram

$$\mathbb{R} \longleftarrow \mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}$$

in **FDlg** has double points at origin. Moreover, the D-topology τ_X is not Hausdorff. One can also show by continuity that the pushout of this diagram in **Man** exists, and it is $\mathbb{R}$. Therefore, $X \ncong \mathbb{R}$ in **FDlg**.

Theorem 2.28 [32] *Let* $G : \mathcal{J} \longrightarrow$ **Man** *be a functor from a small category such that* $\operatorname{colim} G$ *exists in* **Man**. *Then, the canonical smooth map*

$$\eta : \operatorname{colim}(\mathrm{FG_M} \circ G) \longrightarrow \mathrm{FG_M}(\operatorname{colim} G)$$

induces a surjective map

$$|\eta| : |\operatorname{colim}(\mathrm{FG_M} \circ G)| \longrightarrow |\mathrm{FG_M}(\operatorname{colim} G)|.$$

Proof The canonical smooth map $\eta : \operatorname{colim}(\mathrm{FG_M} \circ G) \longrightarrow \mathrm{FG_M}(\operatorname{colim} G)$ comes from the universal property of colimits in **Man** and in **FDlg**.

Assume that $|\eta|$ is not surjective. Say $y \in |\mathrm{FG_M}(\operatorname{colim} G)|$ is not in the image. Then, $A := \operatorname{colim} G \setminus \{y\}$ is a smooth manifold, and $G(j) \longrightarrow \operatorname{colim} G$ factors through $A \hookrightarrow \operatorname{colim} G$ for each $j \in J$ since $|\operatorname{colim}(\mathrm{FG_M} \circ G)| = \operatorname{colim} |\mathrm{FG_M} \circ G|$. Hence, by the universal property of colimit in **Man**, the identity map $\operatorname{colim} G \longrightarrow \operatorname{colim} G$ must factor through $A \hookrightarrow \operatorname{colim} G$, which is impossible. Therefore, $|\eta|$ is surjective. □

Theorem 2.29 [32] *Let* $G : \mathcal{J} \longrightarrow$ **Man** *be a functor from a small category such that* $\operatorname{colim} G$ *exists in* **Man**. *If* $\operatorname{colim}(\mathrm{FG_M} \circ G)$ *is a Frölicher space and its D-topology is Hausdorff, then* $\operatorname{colim}(\mathrm{FG_M} \circ G) \cong \mathrm{FG_M}(\operatorname{colim} G)$.

Proof Let X be a diffeological space. It is straightforward to show that

(1) If $\mathcal{C}^\infty(X, \mathbb{R})$ separates points, i.e.,

$$\forall x \neq x' \in X, \exists l \in \mathcal{C}^\infty(X, \mathbb{R}) : l(x) \neq l(x'),$$

then the D-topology τ_X is Hausdorff.

(2) If the D-topology τ_X is Hausdorff, then any plot $\mathbb{R} \longrightarrow X$ with finite image must be constant.

(3) If any plot $\mathbb{R} \longrightarrow X$ with finite image must be constant and X is Frölicher, then $\mathcal{C}^\infty(X, \mathbb{R})$ separates points.

By Theorem 2.28, the canonical smooth map

$$\eta : \operatorname{colim}(\mathrm{FG_M} \circ G) \longrightarrow \mathrm{FG_M}(\operatorname{colim} G)$$

induces a surjective map

$$|\eta| : |\operatorname{colim}(\mathrm{FG_M} \circ G)| \longrightarrow |\mathrm{FG_M}(\operatorname{colim} G)|.$$

For any $l \in \mathcal{C}^\infty(\operatorname{colim}(\mathrm{FG_M} \circ G), \mathbb{R})$, by the universal property of colimits in **Man** and in **FDlg**, there exists a unique smooth map $f : \operatorname{colim} G \longrightarrow \mathbb{R}$ such that $\mathrm{FG_M}(f) \circ \eta = l$.

Since $\operatorname{colim}(\mathrm{FG_M} \circ G)$ is Frölicher and its D-topology is Hausdorff, from the above we know that $\mathcal{C}^\infty(\operatorname{colim}(\mathrm{FG_M} \circ G), \mathbb{R})$ separates points. Then, the equality $\mathrm{FG_M}(f) \circ \eta = l$ implies that $|\eta|$ is injective.

For any $d \in_U \mathrm{FG_M}(\operatorname{colim} G)$, $l \circ \eta^{-1} \circ d = \mathrm{FG_M}(f) \circ d$ is smooth for any $l \in \mathcal{C}^\infty(\operatorname{colim}(\mathrm{FG_M} \circ G), \mathbb{R})$. Since $\operatorname{colim}(\mathrm{FG_M} \circ G)$ is Frölicher, $\eta^{-1} \circ d \in_U \operatorname{colim}(\mathrm{FG_M} \circ G)$. Hence, η^{-1} is smooth.

Therefore, $\operatorname{colim}(\mathrm{FG_M} \circ G) \cong \mathrm{FG_M}(\operatorname{colim} G)$. □

Here is an immediate application:

Example 2.30 Recall from Example 2.22 that the pushout of

$$\mathbb{R} \xleftarrow{\;0\;} \mathbb{R}^0 \xrightarrow{\;0\;} \mathbb{R}$$

in **FDlg** is the union of the two axes in $\mathbb{R}^2$ with the sub-diffeology, which is a Frölicher space with Hausdorff D-topology, but clearly not a smooth manifold. By Theorem 2.29, the pushout of the above diagram does not exist in **Man**.

2.5 Categorical frameworks for generalized functions

We start by asking the following question: What is a "good" category to frame spaces like $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{A}_q(\Omega)$, $U(\Omega)$, $\mathcal{G}^s(\Omega)$ and $\mathcal{G}^e(\Omega)$?

The list in the following remark permits to restrict the range of choices:

Remark 2.31 (i) Schwartz distribution theory is classically framed using locally convex topological vector spaces (LCTVS), so it is natural to search for a category which contains the category **LCS** of LCTVS and continuous linear maps as a subcategory.

(ii) The space $\mathcal{A}_0(\Omega)$ is an affine space and is usually identified with its underlying vector space (see e.g., [24]). However, it seems that the necessity of this identification is only due to the choice of a category like **LCS**, which is not closed with respect to arbitrary subspaces. It would be better to choose a complete category.

(iii) $U(\Omega)$ and $\mathcal{A}_0(\Omega)$ can be viewed as manifolds modeled in convenient vector spaces (CVS, [13,29]). However, the category of this type of manifolds is not Cartesian closed [29], whereas Cartesian closedness is a basic choice preferred by many mathematicians working with infinite-dimensional spaces (see e.g., [19] and references therein).

(iv) The candidate category shall contain the category $\mathbf{Con}^\infty$ of convenient vector spaces and generic smooth maps [13,29] as a (full) subcategory because the differential calculus of these spaces is used in the study of Colombeau algebras [24]. Note that, we have embeddings $\mathbf{Con} \subseteq \mathbf{LCS} \subseteq \mathbf{Con}^\infty$, where **Con** is the category of CVS and continuous linear maps.

(v) The candidate category must be closed with respect to arbitrary quotient spaces, so as to contain the quotient algebras $\mathcal{G}^s(\Omega)$ and $\mathcal{G}^e(\Omega)$. Of course, a better choice would be to consider a cocomplete category. Since, generally speaking, CVS are not closed with respect to quotient spaces (see [29, page 22]), the candidate category cannot be $\mathbf{Con}^\infty$.

(vi) The candidate category must also contain non-linear maps like the product of GF, e.g.,

$$(u, v) \in \mathcal{G}^s(\Omega) \times \mathcal{G}^s(\Omega) \mapsto u \cdot v \in \mathcal{G}^s(\Omega)$$

or, more generally, any non-linear smooth operation $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ which can be extended to an operation of algebras of GF, e.g.,

$$(u_1, \dots, u_n) \in (\mathcal{G}^s(\Omega))^n \mapsto [f(u_{1\varepsilon}(-), \dots, u_{n\varepsilon}(-))] \in \mathcal{G}^s(\Omega).$$

Another feature of the candidate category we are looking for is to contain as arrows the maps between infinite-dimensional spaces like convolutions, derivatives and integrals of GF.

From literature, we know that two categories satisfying all these requirements: the category **Dlg** of diffeological spaces and the category **Fr** of Frölicher spaces. In the present work, we will also introduce the category **FDlg** of functionally generated spaces as another framework for GF, trying to take the best ideas and properties from both **Dlg** and **Fr**. We will see that all the spaces $\mathcal{D}_K(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{A}_q(\Omega)$, $U(\Omega)$, $\mathcal{G}^s(\Omega)$ and $\mathcal{G}^e(\Omega)$ are objects of (some of) these categories, and in this paper, we in particular study them as (functionally generated) diffeological spaces.

3 Topologies for spaces of generalized functions

Kriegl and Michor [29, page 2] declared that "locally convex topology is not appropriate for non-linear questions in infinite dimensions". Indirectly, this is also confirmed by the fact that topology plays a less important role in categories like **Dlg** or **Fr**. The main aim of this section is to highlight some relationship between Cartesian closedness and locally convex topology.

3.1 Locally convex vector spaces and Cartesian closed categories

The problems that arise in relating locally convex topology and Cartesian closedness can be expressed as follows:

Theorem 3.1 *Let $F \in \mathbf{LCS}$, and let $(\mathcal{T}, U)$ be a Cartesian closed concrete category over* **Top**, *with exponential objects given by the hom-functor $\mathcal{T}(-,-)$ and the forgetful functor $U : \mathcal{T} \longrightarrow$* **Top** *acting as identity on arrows. Assume that $R, \bar{F} \in \mathcal{T}$ such that $U(R) = \mathbb{R}$ and $U(\bar{F}) = F$. Set $F' := \mathbf{LCS}(F, \mathbb{R})$ for the continuous dual of F, and assume that*

$$F' \subseteq |U(\mathcal{T}(\bar{F}, R))|,$$
$$|U(\bar{F} \times \mathcal{T}(\bar{F}, R))| = \left|U(\bar{F}) \times U(\mathcal{T}(\bar{F}, R))\right|, \tag{3.1}$$

and the topology of the space $U(\bar{F} \times \mathcal{T}(\bar{F}, R))$ is coarser than the product topology of $U(\bar{F}) \times U(\mathcal{T}(\bar{F}, R))$. Finally, assume that for every $g \in F'$, the map $(\lambda \in \mathbb{R} \mapsto \lambda \cdot g \in F')$ is continuous with respect to the topology induced on F' by (3.1). Then, the locally convex topology on the space F is normable.

Proof The idea for the proof is only a reformulation of the corresponding result in [29, page 2]. Since $\mathcal{T}$ is Cartesian closed, every evaluation

$$\mathrm{ev}_{XY}(x, f) := f(x) \quad \forall x \in X \, \forall f \in \mathcal{T}(X, Y)$$

is an arrow of $\mathcal{T}$. This is a general result in every Cartesian closed category; see e.g., [2]. Thus, $U(\mathrm{ev}_{XY}) = \mathrm{ev}_{XY}$ is a continuous function. In particular, $\mathrm{ev}_{\bar{F}R} : U(\bar{F} \times \mathcal{T}(\bar{F}, R)) \longrightarrow U(R) = \mathbb{R}$ is continuous. By assumption, also $\mathrm{ev}_{\bar{F}R} : F \times U(\mathcal{T}(\bar{F}, R)) \longrightarrow \mathbb{R}$ is continuous. Therefore, also its restriction to the subspace $F' = \mathbf{LCS}(F, \mathbb{R}) \subseteq |U(\mathcal{T}(\bar{F}, R))|$ is (jointly) continuous:

$$\varepsilon := \mathrm{ev}_{\bar{F}R}|_{F \times F'} : F \times F' \longrightarrow \mathbb{R}.$$

Hence, we can find neighborhoods $U \subseteq F$ and $V \subseteq F'$ of the corresponding zeros such that $\varepsilon(U \times V) \subseteq [-1, 1]$. That is,

$$U \subseteq \{u \in F \mid \forall f \in V : \ |f(u)| \leq 1\}.$$

But then, because the map $(\lambda \in \mathbb{R} \mapsto \lambda \cdot g \in F')$ is continuous, taking a generic functional $g \in F'$, we can always find $\lambda \in \mathbb{R}_{\neq 0}$ such that $\lambda g \in V$, and hence $|g(u)| \leq 1/\lambda$ for every $u \in U$. Any continuous functional is thus bounded on U, so the neighborhood U itself is bounded (see e.g., [27]). Since the topology of any locally convex vector space with a bounded neighborhood of zero is normable (see e.g., [27]), we get the conclusion. □

If, in this theorem, we take $F = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ or $F = \mathcal{D}(\Omega)$ or any other non-normable LCTVS, there are two possibilities to make the space F an object in a Cartesian closed category:

(a) F belongs to a Cartesian closed category $\mathcal{T}$, but $\mathcal{T}$ is not a concrete category over **Top**. This is the solution used in CVS theory in which $\mathcal{T}$ is taken to be the Cartesian closed category $\mathbf{Con}^\infty$. Note that, **LCS** is not a full subcategory of $\mathbf{Con}^\infty$ since not every arrow of $\mathbf{Con}^\infty$ is continuous. A typical example of a $\mathbf{Con}^\infty$-smooth, but not continuous map (with respect to the given locally convex topology instead of the D-topology) is the evaluation

$$\mathrm{ev} : (x, g) \in F \times F' \mapsto g(x) \in \mathbb{R}.$$

(b) For the solution adapted using diffeological spaces, we can take $\mathcal{T} = \mathbf{Dlg}$. But then several assumptions of Theorem 3.1 fail. For example, in general $\tau_{X \times Y} \supseteq \tau_X \times \tau_Y$, but not the opposite as required. Moreover, the D-topology on $\mathcal{D}(\Omega)$ is not normable since it is finer than the usual locally convex topology, which is already not normable. On the contrary, there is no problem verifying the continuity of the scalar multiplication map, as stated in the following:

Theorem 3.2 *Let F be one of the spaces $\mathcal{C}^\infty(\Omega, \mathbb{R})$ and $\mathcal{D}(\Omega)$, and let τ_F be the D-topology on F. Let*

$$F'_{\mathrm{s}} := (\{l \in \mathcal{C}^\infty(F, \mathbb{R}) \mid l \text{ is linear}\} \prec \mathcal{C}^\infty(F, \mathbb{R}))$$

be the smooth dual of F, and let $\tau_{F'_{\mathrm{s}}}$ be the D-topology on F'_{s}. Then, with respect to the pointwise operations, both spaces (F, τ_F) and $(F'_{\mathrm{s}}, \tau_{F'_{\mathrm{s}}})$ are topological vector spaces.

More generally, the D-topology functor always sends a diffeological vector space to a topological vector space; see [42].

Proof We proceed for the case $F = \mathcal{C}^\infty(\Omega, \mathbb{R})$ since the other one is very similar. For simplicity set $Y^X := \mathcal{C}^\infty(X, Y)$, and write

$$\begin{aligned}
&\langle -, - \rangle : (u, v) \in F \times F \mapsto (r \in \Omega \mapsto (u(r), v(r)) \in \mathbb{R}^2) \in (\mathbb{R}^2)^\Omega \\
&\gamma_1 : (u, v) \in (\mathbb{R}^2)^\Omega \times \mathbb{R}^{\mathbb{R}^2} \mapsto v \circ u \in F \\
&\gamma_2 : (u, v) \in \mathbb{R}^\mathbb{R} \times F \mapsto u \circ v \in F \\
&s_\mathbb{R} : (r, s) \in \mathbb{R}^2 \mapsto r + s \in \mathbb{R} \\
&p_\mathbb{R} : (r, s) \in \mathbb{R}^2 \mapsto r \cdot s \in \mathbb{R} \\
&(-, -) : (u, v) \in F^{\mathbb{R}\times F} \times F^{\mathbb{R}\times F} \\
&\mapsto ((\lambda, f) \in \mathbb{R} \times F \mapsto (u(\lambda, f), v(\lambda, f)) \in F \times F) \in (F \times F)^{\mathbb{R}\times F}.
\end{aligned}$$

It is easy to prove that the pointwise sum and pointwise scalar multiplication maps are given by $(-) + (-) = \gamma_1(-, s_\mathbb{R}) \circ \langle -, - \rangle$ and $(-) \cdot (-) = \gamma_2 \circ (p_\mathbb{R}^\wedge \circ q_1, q_2)$, where $q_1 : \mathbb{R} \times F \longrightarrow \mathbb{R}$ and $q_2 : \mathbb{R} \times F \longrightarrow F$ are the projections. Therefore, both sum and scalar multiplication in F are composition or pairing of smooth functions, and hence they are smooth and continuous with respect to the D-topology. Analogously, we can proceed with the smooth dual F'_s by considering the properties of the operator $(- \prec -)$. □

4 Spaces of compactly supported functions as functionally generated spaces

It is very easy to see that the spaces $\mathcal{D}(\Omega) = \{f \in \mathcal{C}^\infty(\Omega, \mathbb{R}) \mid \mathrm{supp}(f) \Subset \Omega\}$ and $\mathcal{D}_K(\Omega) = \{f \in \mathcal{D}(\Omega) \mid \mathrm{supp}(f) \Subset K\}$ with $K \Subset \Omega$ are functionally generated spaces. Recall that $\mathcal{D}_K(\Omega)$ is a LCTVS whose topology is induced by the family of norms (Fréchet structure)

$$\|\phi\|_{K,m} := \max_{|\alpha|\le m} \max_{x\in K} \|\partial^\alpha \phi(x)\| \qquad \forall \phi \in \mathcal{D}_K(\Omega)\ \forall m \in \mathbb{N}. \tag{4.1}$$

Also the space $\mathcal{D}(\Omega)$ is a LCTVS obtained as the inductive limit (i.e., colimit) of $\mathcal{D}_K(\Omega)$ for $K \Subset \Omega$.

Recall that on the space $\mathcal{D}'(\Omega)$ of distributions (i.e., linear maps $l : |\mathcal{D}(\Omega)| \longrightarrow \mathbb{R}$ which are continuous with respect to the locally convex topology), there is a topology called the *weak* topology*, i.e., the coarsest topology such that each evaluation $\mathrm{ev}_\phi : u \in \mathcal{D}'(\Omega) \mapsto \langle u, \phi \rangle \in \mathbb{R}$ is continuous. With respect to this topology, $\mathcal{D}'(\Omega)$ is a LCTVS.

The so-called *canonical diffeology* on these spaces is a particular case of the following:

Definition 4.1 Let V be a topological vector space. The *canonical diffeology* $\mathbb{D}(V) = \bigcup_{U\in\mathcal{O}\mathbb{R}^\infty} \mathbb{D}_U(V)$ is given by the sets $\mathbb{D}_U(V)$ of all maps $d : U \longrightarrow V$ which are smooth when tested by continuous linear functionals, i.e.,

$$\forall l : V \longrightarrow \mathbb{R} \text{ continuous linear: } l \circ d \in \mathcal{C}^\infty(U, \mathbb{R}).$$

Therefore, (see Definition 2.17 and Remark 2.18), $(V, \mathbb{D}(V)) \in$ **FDlg**. Since the functionals are globally defined, this is also a Frölicher space, i.e.,

$$\mathrm{D_F}(\mathrm{F_D}(V, \mathbb{D}(V))) = (V, \mathbb{D}(V)).$$

We will continue to denote the spaces by $\mathcal{D}(\Omega)$ and $\mathcal{D}_K(\Omega)$ even when we think of them as diffeological spaces with the canonical diffeology. When we want to underscore that we are considering them only as LCTVS, we will use the notations $\mathcal{D}^{\mathrm{LC}}(\Omega)$ and $\mathcal{D}_K^{\mathrm{LC}}(\Omega)$.

4.1 Plots of $\mathcal{D}_K(\Omega)$, $\mathcal{D}(\Omega)$ and Cartesian closedness

It is also interesting to reformulate the property of being a plot $d \in_U \mathcal{D}(\Omega)$ (or $d \in_U \mathcal{D}_K(\Omega)$) using Cartesian closedness. This permits to compare better the canonical diffeology on these spaces from LCTVS with the diffeology induced on them as subspaces of $\mathcal{C}^\infty(\Omega, \mathbb{R})$. We will denote by $\mathcal{D}^\mathrm{s}(\Omega)$ this latter diffeological space. So $d \in_U \mathcal{D}^\mathrm{s}(\Omega)$ if and only if $d : U \longrightarrow |\mathcal{D}(\Omega)|$ and $i \circ d \in_U \mathcal{C}^\infty(\Omega, \mathbb{R})$, where $i : \mathcal{D}(\Omega) \hookrightarrow \mathcal{C}^\infty(\Omega, \mathbb{R})$ is the inclusion. Recall that $i \circ d \in_U \mathcal{C}^\infty(\Omega, \mathbb{R})$ if and only if $(i \circ d)^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$. Analogously, we define $\mathcal{D}_K^\mathrm{s}(\Omega)$. In performing the comparison, we will use Lem. 2.1, Lem. 2.2 and Thm. 2.3 of [28] which are cited here for reader's convenience. In this subsection, without confusion we use the same notation for morphisms in different categories when the functions for the underlying sets are the same.

Lemma 4.2 (2.1 of [28]) *If* $U \in \mathcal{O}\mathbb{R}^\infty$ *and* $f \in \mathcal{C}^\infty(U, \mathcal{D}(\Omega))$*, then* $f : U \longrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega)$ *is continuous.*

To state the other cited results of [28], we need the following:

Definition 4.3 Let $U \in \mathcal{O}\mathbb{R}^\infty$ and let $f : U \times \Omega \longrightarrow \mathbb{R}$ be a map. We say that

$$f \text{ is of uniformly bounded support (with respect to } U)$$

if

$$\exists K \Subset \Omega \,\forall u \in U : \ \mathrm{supp}(f(u, -)) \subseteq K.$$

We say that

$$f \text{ is locally of uniformly bounded support}$$

if

$$\forall u \in U \,\exists V \text{ open neigh. of } u \text{ in } U : \ f|_{V\times\Omega} \text{ is of uniformly bounded support.}$$

Finally, we say that

$$f \text{ is pointwise of bounded support}$$

if

$$\forall u \in U \,\exists K \Subset \Omega : \ \mathrm{supp}(f(u, -)) \subseteq K.$$

Using this definition, we can state

Lemma 4.4 (2.2 of [28]) *Let* $U \in \mathcal{O}\mathbb{R}^\infty$ *and assume that* $f \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ *is pointwise of bounded support. Then, the following are equivalent*

(i) f *is locally of uniformly bounded support;*
(ii) $f^\wedge : U \longrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega)$ *is continuous.*

Theorem 4.5 (2.3 of [28]) *Let* $U \in \mathcal{O}\mathbb{R}^\infty$*. Then, the following are equivalent:*

(i) $f \in \mathcal{C}^\infty(U, \mathcal{D}(\Omega))$*;*
(ii) $f^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ *and* $f^\vee$ *is locally of uniformly bounded support.*

In other words, Theorem 4.5 says that $d \in_U \mathcal{D}(\Omega)$ if and only if $d \in_U \mathcal{D}^{\mathrm{s}}(\Omega)$ and $d^\vee$ is locally of uniformly bounded support, and hence we have $\mathcal{D}(\Omega) \subseteq \mathcal{D}^{\mathrm{s}}(\Omega)$.

From these results, we can also solve the same comparison problem for the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}^{\mathrm{s}}_K(\Omega)$. The following lemma is analogous to Lemma 4.2 for $\mathcal{D}_K(\Omega)$.

Lemma 4.6 *If* $f \in \mathcal{C}^\infty(U, \mathcal{D}_K(\Omega))$ *with* $U \in \mathcal{O}\mathbb{R}^\infty$ *and* $K \Subset \Omega$*, then* $f : U \longrightarrow \mathcal{D}^{\mathrm{LC}}_K(\Omega)$ *is continuous.*

Proof Since the inclusion map $i_K : \mathcal{D}^{\mathrm{LC}}_K(\Omega) \hookrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega)$ is continuous linear, by post-composition it takes continuous linear functionals $l : \mathcal{D}^{\mathrm{LC}}(\Omega) \longrightarrow \mathbb{R}$ to continuous linear functionals $l \circ i_K : \mathcal{D}^{\mathrm{LC}}_K(\Omega) \longrightarrow \mathbb{R}$. From Theorem 2.11 it follows that $i_K \in \mathcal{C}^\infty(\mathcal{D}_K(\Omega), \mathcal{D}(\Omega))$, and hence $i_K \circ f \in \mathcal{C}^\infty(U, \mathcal{D}(\Omega))$. Therefore, Lemma 4.2 implies that $i_K \circ f$ is continuous and hence the conclusion since the topology on $\mathcal{D}^{\mathrm{LC}}_K(\Omega)$ coincides with the initial topology induced by i_K. □

The following lemma is analogous to Lemma 4.4 for $\mathcal{D}_K(\Omega)$.

Lemma 4.7 *Let* $U \in \mathcal{O}\mathbb{R}^\infty$ *and let* $f \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$*. If there exists* $K \Subset \Omega$ *such that*

$$\forall u \in U : \ supp(f(u, -)) \subseteq K, \tag{4.2}$$

then $f^\wedge : U \longrightarrow \mathcal{D}^{\mathrm{LC}}_K(\Omega)$ *is continuous.*

Proof Clearly, f is locally of uniformly bounded support. Apply Lemma 4.4, we know that $i_K \circ f^\wedge : U \longrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega)$ is continuous. Since $\mathcal{D}^{\mathrm{LC}}_K(\Omega)$ has the initial topology from $i_K : \mathcal{D}^{\mathrm{LC}}_K(\Omega) \hookrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega)$, $f^\wedge : U \longrightarrow \mathcal{D}^{\mathrm{LC}}_K(\Omega)$ is continuous. □

Finally, the following theorem is analogous to Theorem 4.5 for $\mathcal{D}_K(\Omega)$.

Theorem 4.8 *Let $U \in \mathcal{O}\mathbb{R}^\infty$ and let $K \Subset \Omega$. Then, the following are equivalent:*

(i) $f \in \mathcal{C}^\infty(U, \mathcal{D}_K(\Omega))$;
(ii) $f^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ *and* $supp(f^\vee(u, -)) \subseteq K$ *for every* $u \in U$.

Proof (i) $\Rightarrow$ (ii). We already proved in Lemma 4.6 that the inclusion map $i_K \in \mathcal{C}^\infty(\mathcal{D}_K(\Omega), \mathcal{D}(\Omega))$, so $i_K \circ f \in \mathcal{C}^\infty(U, \mathcal{D}(\Omega))$. By Theorem 4.5 we have $(i_K \circ f)^\vee = f^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$. The second part of the conclusion follows from the definition of the codomain $\mathcal{D}_K(\Omega)$ of f in (i).

(ii) $\Rightarrow$ (i). Assumption (ii) implies that f is locally of uniformly bounded support. From Theorem 4.5 we thus obtain that $f \in \mathcal{C}^\infty(U, \mathcal{D}(\Omega))$. But the assumption implies that $f(U) \subseteq |\mathcal{D}_K(\Omega)|$. So, the conclusion follows from the following Lemma:

Lemma 4.9 *If $K \Subset \Omega$, then $(|\mathcal{D}_K(\Omega)| \prec \mathcal{D}(\Omega)) = \mathcal{D}_K(\Omega)$.*

Proof We have to prove that figures of the spaces on both sides are equal.
$(|\mathcal{D}_K(\Omega)| \prec \mathcal{D}(\Omega)) \supseteq \mathcal{D}_K(\Omega)$: This follows directly from the fact that the inclusion map i_K is in $\mathcal{C}^\infty(\mathcal{D}_K(\Omega), \mathcal{D}(\Omega))$.
$(|\mathcal{D}_K(\Omega)| \prec \mathcal{D}(\Omega)) \subseteq \mathcal{D}_K(\Omega)$: Assume that $d : U \longrightarrow \mathcal{D}_K(\Omega)$ is a map such that $i_K \circ d \in_U \mathcal{D}(\Omega)$, i.e., $\lambda \circ i_K \circ d \in \mathcal{C}^\infty(U, \mathbb{R})$ for every $\lambda \in \mathcal{D}'(\Omega)$. We need to prove that $l \circ d \in \mathcal{C}^\infty(U, \mathbb{R})$ for every continuous linear maps $l : \mathcal{D}^{\mathrm{LC}}_K(\Omega) \longrightarrow \mathbb{R}$. So the problem is to extend any such given l to some $\lambda \in \mathcal{D}'(\Omega)$. To this end, we can repeat the usual proof of the local form of distributions as derivatives of continuous functions to obtain the following:

Theorem 4.10 *For any continuous linear map $l : \mathcal{D}^{\mathrm{LC}}_K(\Omega) \longrightarrow \mathbb{R}$, there exist $g \in \mathcal{C}^0(\Omega, \mathbb{R})$ and $\alpha \in \mathbb{N}^n$ such that*

$$l(\phi) = \langle \partial^\alpha g, \phi \rangle \qquad \forall \phi \in |\mathcal{D}_K(\Omega)|.$$

Therefore, the continuous linear functional $\langle \partial^\alpha g, - \rangle : \mathcal{D}^{\mathrm{LC}}(\Omega) \longrightarrow \mathbb{R}$ extends the functional l.

The conclusion then follows by applying this theorem.

Corollary 4.11 *If $K \Subset \Omega$, then $\mathcal{D}_K(\Omega) = \mathcal{D}^{\mathrm{s}}_K(\Omega)$.*

Fact. Indeed Theorem 4.8 says that $d \in_U \mathcal{D}_K(\Omega)$ if and only if $d \in_U \mathcal{D}^{\mathrm{s}}_K(\Omega)$.

4.2 The locally convex topology and the D-topology on $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

In this subsection, we present some results about linear functionals on the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$ which are continuous with respect to the locally convex topology (or the D-topology). The first result follows at once from Lemmas 4.2 and 4.6:

Corollary 4.12 *On the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$, the D-topology is finer than the locally convex topology.*

It remains open whether the D-topology is strictly finer than the locally convex topology or not. We first study the behavior of the maps of the form $\lambda : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega')$, where $\Omega' \subseteq \mathbb{R}^d$ is open, and henceforth we always assume this.

Theorem 4.13 (i) *$\mathcal{D}(\Omega)$ is a CVS.*
(ii) *If $T \in \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})$ is linear, then $T : \mathcal{D}^{\mathrm{LC}}(\Omega) \longrightarrow \mathbb{R}$ is continuous.*

The same results hold for $\mathcal{D}_K(\Omega)$.

Proof See [28, page 5, 6, 9] or [29, Lem. 6.2, page 67].

The following lemma is a trivial consequence of Theorem 2.11, but we prefer to state it here for completeness.

Lemma 4.14 *If* $\lambda : \mathcal{D}^{\mathrm{LC}}(\Omega) \longrightarrow \mathcal{D}^{\mathrm{LC}}(\Omega')$ *is continuous linear, then* $\lambda \in \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathcal{D}(\Omega'))$.

In the following results, we show that if a linear map $|\mathcal{D}(\Omega)| \longrightarrow \mathbb{R}$ is D-continuous (i.e., continuous with respect to the D-topology), then it is a distribution:

Theorem 4.15 *If* $l : T_D(\mathcal{D}(\Omega)) \longrightarrow \mathbb{R}$ *is continuous linear, then*

$$l \in \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R}) \cap \mathcal{D}'(\Omega).$$

The schema to prove this theorem is the following: we need to prove that $l \circ d \in \mathcal{C}^\infty(U, \mathbb{R})$ whenever $d \in_U \mathcal{D}(\Omega)$, i.e., by Theorem 4.5, if $d^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ and $d^\vee$ is locally of uniformly bounded support. We are going to prove that:

(i) For any $u \in U$ the limit

$$\lim_{h\to 0} \frac{d(u + he_i) - d(u)}{h} \tag{4.3}$$

exists in $T_D(\mathcal{D}(\Omega))$, where

$$e_i = (0, \overset{i-1}{\ldots\ldots}, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \supseteq U.$$

In fact, this limit is $(\frac{\partial d^\vee}{\partial e_i})^\wedge$, which is again a figure of type U of $\mathcal{D}(\Omega)$.

(ii) Since $l : T_D(\mathcal{D}(\Omega)) \longrightarrow \mathbb{R}$ is continuous linear, we can apply (i) and commute l with the limit and the incremental ratio to prove that $\frac{\partial}{\partial e_i}(l \circ d)$ exists and is of the form $l \circ p$ with $p \in_U \mathcal{D}(\Omega)$. The conclusion then follows by induction.

Before proving (i), it is indispensable to have the following:

Lemma 4.16 *Let* V *be an open set in* $\mathbb{R}^n$. *Then, the spaces* $T_D(\mathcal{C}^\infty(V, \mathbb{R}))$ *and* $T_D(\mathcal{D}(V))$ *are Hausdorff.*

More generally, one can show by a similar method that if X and Y are diffeological spaces such that $T_D(Y)$ is Hausdorff. Then, $T_D(C^\infty(X, Y))$ is also Hausdorff.

Proof Note that for any $v \in V$, the evaluation maps $l_v : h \in \mathcal{C}^\infty(V, \mathbb{R}) \mapsto h(v) \in \mathbb{R}$ and $\bar{l}_v : h \in \mathcal{D}(V) \mapsto h(v) \in \mathbb{R}$ are smooth, and hence the maps $T_D(l_v) : T_D(\mathcal{C}^\infty(V, \mathbb{R})) \longrightarrow \mathbb{R}$ and $T_D(\bar{l}_v) : T_D(\mathcal{D}(V)) \longrightarrow \mathbb{R}$ are both continuous. Therefore, the functional topology on $\mathcal{C}^\infty(V, \mathbb{R})$ and $\mathcal{D}(V)$ is Hausdorff. The conclusion then follows from Theorem 2.13. □

We now prove Theorem 4.15:

Proof To prove the existence of the limit in (i), we first fix $d \in_U \mathcal{D}(\Omega)$, $u \in U$, $e_i = (0, \overset{i-1}{\ldots\ldots}, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \supseteq U$ and $r \in \mathbb{R}_{>0}$ such that $B_r(u) \subseteq U$. Then, there exist an open neighborhood V of u in U and $a \in \mathbb{R}_{>0}$ such that $v + he_i \in B_r(u)$ for all $v \in V$ and $h \in (-a, a)$. Set $H := (-a, a)$, and for any $h \in H$, define

$$\delta(h) := \left((v, x) \in V \times \Omega \mapsto \int_0^1 \frac{\partial d^\vee}{\partial e_i}(v + she_i, x)\, \mathrm{d}s \in \mathbb{R} \right).$$

Clearly $\delta(0) = \frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega}$. Theorem 4.5 implies $d^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$, so $\delta^\vee \in \mathcal{C}^\infty(H \times V \times \Omega, \mathbb{R})$, and hence $\delta \in_H \mathcal{C}^\infty(V \times \Omega, \mathbb{R}) =: \mathbb{R}^{V\times\Omega}$. Also note that for any non-zero $h \in H$ and for any $(v, x) \in V \times \Omega$, by the fundamental theorem of calculus, we have

$$\delta^\vee(h, v, x) = \frac{d^\vee(v + he_i, x) - d^\vee(v, x)}{h}. \tag{4.4}$$

We prove below that $\lim_{h\to 0} \delta(h) = \frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega}$ in the space $\mathbb{R}^{V\times(\Omega)}$, which has the underlying set

$$|\mathbb{R}^{V\times(\Omega)}| := \{\phi \in \mathbb{R}^{V\times\Omega} \mid \phi^\wedge \in_V \mathcal{D}(\Omega)\},$$

and figures defined by $p \in_W \mathbb{R}^{V\times(\Omega)}$ iff $p^\vee : W \times V \times \Omega \longrightarrow \mathbb{R}$ is smooth and locally of uniformly bounded support with respect to $W \times V$.

Since $d^\vee$ is locally of uniformly bounded support (Theorem 4.5), we may assume that V and H are sufficiently small so that $\delta^\vee : H \times V \times \Omega \longrightarrow \mathbb{R}$ is of uniformly bounded support with respect to $H \times V$. Thus,

$$\delta \in_H \mathbb{R}^{V\times(\Omega)}. \tag{4.5}$$

To prove the above-mentioned limit equality, let A be a D-open subset of $\mathbb{R}^{V\times(\Omega)}$ such that $\frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega} \in A$. From (4.5) we know that $\delta^{-1}(A) =: B$ is open in H. Moreover, $\delta(0) = \frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega} \in A$ implies $0 \in B$. This proves that $\lim_{h\to 0}\delta(h) = \frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega}$ in $\mathbb{R}^{V\times(\Omega)}$.

Now, we apply this limit to the adjoint map

$$(-)^\wedge : \phi \in |\mathbb{R}^{V\times(\Omega)}| \mapsto \phi^\wedge \in |\mathcal{D}(\Omega)^V|, \tag{4.6}$$

where the domain is the diffeological space $\mathbb{R}^{V\times(\Omega)}$, and the codomain is the space $\mathcal{D}(\Omega) \uparrow V$ with $|\mathcal{D}(\Omega)^V| = |\mathcal{C}^\infty(V, \mathcal{D}(\Omega))|$ as the underlying set and figures defined by $q \in_{\tilde{W}} \mathcal{D}(\Omega) \uparrow V$ iff $(q^\vee)^\vee : \tilde{W} \times V \times \Omega \longrightarrow \mathbb{R}$ is smooth and locally of uniformly bounded support. We claim that the adjoint map (4.6) is smooth with respect to these diffeological structures on its domain and codomain. In fact, if $p \in_W \mathbb{R}^{V\times(\Omega)}$, then $(((-)^\wedge \circ p)^\vee)^\vee = p^\vee$, which is locally of uniformly bounded support by the definition of the diffeology on $\mathbb{R}^{V\times(\Omega)}$. Therefore, $(-)^\wedge : \mathbb{R}^{V\times(\Omega)} \longrightarrow \mathcal{D}(\Omega) \uparrow V$ is smooth, and hence it is D-continuous:

$$\left(\frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega}\right)^\wedge = \left(\lim_{h\to 0}\delta(h)\right)^\wedge = \lim_{h\to 0}\delta(h)^\wedge \qquad \text{in } \mathcal{D}(\Omega) \uparrow V.$$

Now consider the evaluation at $v \in V \subseteq U$:

$$\mathrm{ev}_v : \phi \in |\mathcal{D}(\Omega) \uparrow V| = |\mathcal{D}(\Omega)^V| \mapsto \phi(v) \in |\mathcal{D}(\Omega)|.$$

We claim that $\mathrm{ev}_v : \mathcal{D}(\Omega) \uparrow V \longrightarrow \mathcal{D}(\Omega)$ is smooth. In fact, $q \in_{\tilde{W}} \mathcal{D}(\Omega) \uparrow V$ means that

$$(q^\vee)^\vee : \tilde{W} \times V \times \Omega \longrightarrow \mathbb{R} \quad \text{is smooth and locally of uniformly bounded support.} \tag{4.7}$$

We need to prove that $(\mathrm{ev}_v \circ q)^\vee : \tilde{W} \times \Omega \longrightarrow \mathbb{R}$ is also smooth and locally of uniformly bounded support. Take $w \in \tilde{W}$. Then by (4.7) there exist open neighborhoods C of w and D of v such that $(q^\vee)^\vee|_{C\times D\times\Omega}$ is of uniformly bounded support. We may assume that $\mathrm{supp}[(q^\vee)^\vee(w', v', -)] \subseteq K \Subset \Omega$ for all $(w', v') \in C \times D$. But $(q^\vee)^\vee(w', v', -) = q(w')(v') = \mathrm{ev}_{v'}(q(w'))$. Therefore, for all $w' \in C$ we have $\mathrm{supp}[(\mathrm{ev}_v \circ q)^\vee(w', -)] = \mathrm{supp}[q(w')(v)] \subseteq K$. By Cartesian closedness, $\mathrm{ev}_v \circ q$ is smooth, and hence it is a figure of $\mathcal{D}(\Omega)$. This proves that $\mathrm{ev}_v : \mathcal{D}(\Omega) \uparrow V \longrightarrow \mathcal{D}(\Omega)$ is smooth, and hence it is D-continuous. So we have:

$$\begin{aligned}\frac{\partial d^\vee}{\partial e_i}(v, -) &= \mathrm{ev}_v\left[\left(\frac{\partial d^\vee}{\partial e_i}|_{V\times\Omega}\right)^\wedge\right] = \mathrm{ev}_v[\lim_{h\to 0}\delta(h)^\wedge] \\ &= \lim_{h\to 0}\delta(h)^\wedge(v) = \lim_{h\to 0}\frac{d^\vee(v + he_i, -) - d^\vee(v, -)}{h} \\ &= \lim_{h\to 0}\frac{d(v + he_i) - d(v)}{h} \qquad \forall v \in V.\end{aligned}$$

Therefore, this limit exists in $\mathcal{D}(\Omega)$. By assumption, $l : |\mathcal{D}(\Omega)| \longrightarrow \mathbb{R}$ is linear and D-continuous, so

$$\begin{aligned} l\left(\frac{\partial d^\vee}{\partial e_i}(v, -)\right) &= l\left[\lim_{h\to 0}\frac{d(v + he_i) - d(v)}{h}\right] \\ &= \lim_{h\to 0}\frac{l(d(v + he_i)) - l(d(v))}{h} \\ &= \frac{\partial(l \circ d)}{\partial e_i}(v).\end{aligned}$$

This proves that the first partial derivatives of $l \circ d$ exist and are D-continuous because both l and $\frac{\partial d^\vee}{\partial e_i}$ are D-continuous. We can now apply the same procedure to the figure

$$\left(\frac{\partial d^\vee}{\partial e_i}\right)^\wedge \in_U \mathcal{D}(\Omega)$$

and conclude that also the second partial derivatives of $l \circ d$ exist and are D-continuous. By applying inductively this process, we get the conclusion $l \circ d \in \mathcal{C}^\infty(U, \mathbb{R})$. Finally, from Theorem 4.13 we have $l \in \mathcal{D}'(\Omega)$. □

As a consequence, we have the following:

Corollary 4.17 *Let $l : |\mathcal{D}(\Omega)| \longrightarrow \mathbb{R}$ be a linear map. Then, the following are equivalent:*

(i) *l is continuous with respect to the locally convex topology on $\mathcal{D}(\Omega)$, i.e., it is a distribution.*
(ii) *l is continuous with respect to the D-topology on $\mathcal{D}(\Omega)$.*
(iii) *$l \in \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})$.*

Proof (i) ⇒ (ii): From Corollary 4.12; (ii) ⇒ (iii): From Theorem 4.15; (iii) ⇒ (i): From Theorem 4.13. □

From the proof of Theorem 4.15, we have

Corollary 4.18 *Let U be an open set in $\mathbb{R}^n$ and let $d \in_U \mathcal{D}(\Omega)$. Then, d is smooth in the usual sense, i.e., for all $\alpha \in \mathbb{N}^n$, the partial derivative $\partial^\alpha d : U \longrightarrow |\mathcal{D}(\Omega)|$ exists as the limit of a suitable incremental ratio in the topological vector space $|\mathcal{D}(\Omega)|$ with the D-topology. Moreover, $\partial^\alpha d \in_U \mathcal{D}(\Omega)$.*

By applying this result to a curve $d \in_{\mathbb{R}} \mathcal{D}(\Omega)$, and knowing that the D-topology is finer than the usual locally convex topology, we get an independent proof that $\mathcal{D}(\Omega)$ is a CVS.

We close this section with the following result, which underscores the possible difference between $\mathcal{D}(\Omega)$ and its counterpart $\mathcal{D}^{\mathrm{s}}(\Omega)$. In the statement, if $F \in \mathbf{Dlg}$ is also a vector space, then we set

$$F'_{\mathrm{s}} := (\{l \in \mathcal{C}^\infty(F, \mathbb{R}) \mid l \text{ is linear}\} \prec \mathcal{C}^\infty(F, \mathbb{R}))$$

for its smooth dual space (this notation has been used for the special cases in Theorem 3.2).

Corollary 4.19 (i) *$|\mathcal{D}'(\Omega)| = |\mathcal{D}(\Omega)'_{\mathrm{s}}|$ and $\mathcal{D}'(\Omega) \supseteq \mathcal{D}(\Omega)'_{\mathrm{s}}$.*
(ii) *$|\mathcal{D}'(\Omega)| \supseteq |\mathcal{D}^{\mathrm{s}}(\Omega)'_{\mathrm{s}}| = \{l \in \mathcal{C}^\infty(\mathcal{D}^{\mathrm{s}}(\Omega), \mathbb{R}) \mid l$ is linear$\}$.*

Proof (i): We first prove that the underlying sets are equal, i.e., $|\mathcal{D}'(\Omega)| = |\mathcal{D}(\Omega)'_{\mathrm{s}}|$. In fact, this follows from the equivalence (i) $\iff$ (iii) of Corollary 4.17. Now, if $d \in_U \mathcal{D}(\Omega)'_{\mathrm{s}}$, then $d^\vee : U \times \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ is smooth. The space $\mathcal{D}'(\Omega)$ is functionally generated by all linear functionals $l : |\mathcal{D}'(\Omega)| \longrightarrow \mathbb{R}$ which are continuous with respect to the weak* topology. Since each of these functionals is of the form $l = \mathrm{ev}_\phi$ for some $\phi \in \mathcal{D}(\Omega)$, we only need to consider $(\mathrm{ev}_\phi \circ d)(u) = \mathrm{ev}_\phi[d(u)] = d(u)(\phi) = d^\vee(u, \phi)$ for every $u \in U$. Therefore, $l \circ d = \mathrm{ev}_\phi \circ d = d^\vee(-, \phi)$ is smooth, which implies that $d \in_U \mathcal{D}'(\Omega)$.

(ii): As a consequence of Theorem 4.5, we know that $\mathcal{D}(\Omega) \subseteq \mathcal{D}^{\mathrm{s}}(\Omega)$. Therefore, if $l \in \mathcal{C}^\infty(\mathcal{D}^{\mathrm{s}}(\Omega), \mathbb{R})$ is linear, then we also have $l \in \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})$. Now $l \in |\mathcal{D}'(\Omega)|$ follows from Corollary 4.17.

5 Spaces for Colombeau generalized functions as diffeological spaces

It is natural to view all the spaces used to define CGF as diffeological spaces. We will start with $\mathcal{C}^\infty(\Omega)^I$, $\mathcal{E}^s_M(\Omega)$, $\mathcal{A}_q(\Omega)$, $U(\Omega)$, $\mathcal{E}^e(\Omega)$ and $\mathcal{E}^e_M(\Omega)$, with the aim to prove that the quotient spaces $\mathcal{G}^s(\Omega)$, $\mathcal{G}^e(\Omega)$ are smooth differential algebras.

The space $\mathcal{C}^\infty(\Omega)^I$

Elements (u_ε) of $\mathcal{C}^\infty(\Omega)^I$ are arbitrary nets, indexed in $\varepsilon \in I$, of smooth functions on Ω. There are studies of Colombeau-like algebras with smooth or continuous ε-dependence (see [7,22] and references therein). In [21], it has been proved that a very large class of equations have no solution if we request continuous dependence with respect to $\varepsilon \in I$. For this reason, it is natural to think of I as a space with the discrete diffeology (see (ii) of Remark 2.3). That is, only locally constant maps $d : U \longrightarrow I$ are figures of I. With this structure, the space I is functionally generated by $\mathbf{Set}(I, \mathbb{R})$. If we think of $\mathcal{C}^\infty(\Omega)$ as the space $\mathcal{C}^\infty(\Omega, \mathbb{R}) \in \mathbf{Dlg}$, then by Cartesian closedness (Theorem 2.24) we have $u \in \mathcal{C}^\infty(\Omega)^I := \mathcal{C}^\infty(I, \mathcal{C}^\infty(\Omega, \mathbb{R}))$ iff $u \in \mathbf{Set}(I, \mathcal{C}^\infty(\Omega))$. The space $\mathcal{C}^\infty(\Omega)^I$ with this diffeological structure will be denoted by $\mathcal{C}^\infty(\Omega, \mathbb{R})^I$. Figures $d \in_U \mathcal{C}^\infty(\Omega, \mathbb{R})^I$ are the maps $d : U \longrightarrow \mathbf{Set}(I, \mathcal{C}^\infty(\Omega))$ such that $(d^\vee)^\vee(-, \varepsilon, -) \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ for every $\varepsilon \in I$.

The space $\mathcal{E}^s_M(\Omega)$

The natural diffeology on

$$\mathcal{E}^s_M(\Omega) = \left\{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall\alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\right\}$$

is the sub-diffeology of $\mathcal{C}^\infty(\Omega, \mathbb{R})^I$. That is,

$$\boldsymbol{\mathcal{E}}^s_M(\Omega) := (\mathcal{E}^s_M(\Omega) \prec \mathcal{C}^\infty(\Omega, \mathbb{R})^I).$$

More precisely, its figures $d \in_U \boldsymbol{\mathcal{E}}^s_M(\Omega)$ are the maps $d : U \longrightarrow \mathcal{E}^s_M(\Omega)$ such that $(d^\vee)^\vee(-, \varepsilon, -) \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ for every $\varepsilon \in I$.

The space $\mathcal{A}_q(\Omega)$

The set $\mathcal{A}_0(\Omega) = \{\phi \in |\mathcal{D}(\Omega)| \mid \int \phi = 1\}$ has a natural diffeology, the sub-diffeology of $\mathcal{D}(\Omega)$. So

$$\boldsymbol{\mathcal{A}}_0(\Omega) := (\mathcal{A}_0(\Omega) \prec \mathcal{D}(\Omega)) \in \mathbf{Dlg}.$$

Analogously, the set

$$\mathcal{A}_q(\Omega) = \left\{\phi \in \mathcal{A}_0(\Omega) \mid \forall\alpha \in \mathbb{N}^n :\ 1 \le |\alpha| \le q \Rightarrow \int x^\alpha \phi(x)\, \mathrm{d}x = 0\right\}$$

has a natural diffeology, the sub-diffeology of $\boldsymbol{\mathcal{A}}_0(\Omega)$. So

$$\boldsymbol{\mathcal{A}}_q(\Omega) := (\mathcal{A}_q(\Omega) \prec \boldsymbol{\mathcal{A}}_0(\Omega)) = (\mathcal{A}_q(\Omega) \prec \mathcal{D}(\Omega)) \in \mathbf{Dlg},$$

where we used the property $(S \prec (T \prec X)) = (S \prec X)$ if $S \subseteq T \subseteq |X|$ and $X \in \mathbf{Dlg}$. Therefore, figures $d \in_U \boldsymbol{\mathcal{A}}_q(\Omega)$ are the maps $d : U \longrightarrow \mathcal{A}_q(\Omega)$ such that $d^\vee \in \mathcal{C}^\infty(U \times \Omega, \mathbb{R})$ and $d^\vee$ is locally of uniformly bounded support (Theorem 4.5).

Note that $\int_\Omega : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ is a smooth linear map. Therefore, $\mathcal{A}_0(\Omega)$ is an affine space which is closed in the locally convex topology of $\mathcal{D}(\Omega)$. An isomorphism with the corresponding vector space $\mathcal{A}_{00}(\Omega) := \ker(\int_\Omega)$ is given by $\phi \in \mathcal{A}_0(\Omega) \mapsto \phi - \phi_0 \in \mathcal{A}_{00}(\Omega)$, where $\phi_0 \in \mathcal{A}_0(\Omega)$ is any fixed element. This isomorphism is clearly a diffeomorphism in **Dlg**. This solves the problem stated in (ii) of Remark 2.31.

The space $U(\Omega)$

In Definition 1.2 of the full Colombeau algebra, the set $U(\Omega) \subseteq \mathcal{A}_0 \times \Omega$ serves as the domain of the representatives $R : U(\Omega) \longrightarrow \mathbb{R}$ of CGF in $\mathcal{G}^e(\Omega)$. These representatives are requested to be smooth in the Ω slot, but with no particular regularity in the $\mathcal{A}_0$ slot. Note that $\mathcal{A}_0$ serves as an index set for the full Colombeau algebra, analogous to the interval I as an index set for the special one. This suggests that we shall equip the discrete diffeology on $\mathcal{A}_0$ and the standard diffeology on Ω. If we identify the set $\mathcal{A}_0$ with the corresponding discrete diffeological space, then

$$\begin{aligned}\boldsymbol{U}(\Omega) &:= (\{(\phi, x) \in \mathcal{A}_0 \times \Omega \mid \mathrm{supp}(\phi) \subseteq \Omega - x\} \prec \mathcal{A}_0 \times \Omega)\\ &= (U(\Omega) \prec |\mathcal{D}(\mathbb{R}^n)| \times \mathbb{R}^n) \in \mathbf{Dlg},\end{aligned}$$

where we used the property $(A \prec D) \times (O \prec Q) = (A \times O \prec D \times Q)$ in **Dlg**, and $|\mathcal{D}(\mathbb{R}^n)|$ is viewed as a discrete diffeological space. Therefore, figures $d \in_V \boldsymbol{U}(\Omega)$ are the maps $d : V \longrightarrow U(\Omega)$ such that the two projections verify $d_1 \in \mathbf{Set}(V, |\mathcal{D}(\mathbb{R}^n)|)$ and $d_2 \in \mathcal{C}^\infty(V, \mathbb{R}^n)$.

The space $\mathcal{E}^e(\Omega)$

The space $\mathcal{E}^e(\Omega)$ (see Definition 1.2) inherits its diffeological structure from $\mathcal{C}^\infty(\boldsymbol{U}(\Omega), \mathbb{R}) \in \mathbf{Dlg}$:

$$\boldsymbol{\mathcal{E}}^e(\Omega) := (\mathcal{E}^e(\Omega) \prec \mathcal{C}^\infty(\boldsymbol{U}(\Omega), \mathbb{R})).$$

Figures $d \in_V \boldsymbol{\mathcal{E}}^e(\Omega)$ are the maps $d : V \longrightarrow \mathcal{E}^e(\Omega)$ such that $d^\vee \in \mathcal{C}^\infty(V \times \boldsymbol{U}(\Omega), \mathbb{R})$. We give an equivalent characterization of $\boldsymbol{\mathcal{E}}^e(\Omega)$ as follows. For $\phi \in \mathcal{A}_0$, set

$$\Omega_\phi := \Omega \cap \{x \in \mathbb{R}^n \mid \operatorname{supp}(\phi) \subseteq \Omega - x\}.$$

As a convention, when $\Omega_\phi = \emptyset$, we think of $\mathcal{C}^\infty(\Omega_\phi, \mathbb{R})$ as a set with a single element. Since $R \in \mathcal{E}^e(\Omega)$ iff $R^\wedge : \mathcal{A}_0 \longrightarrow \bigcup_{\phi\in\mathcal{A}_0} \mathcal{C}^\infty(\Omega_\phi, \mathbb{R})$ and $R(\phi, -) \in \mathcal{C}^\infty(\Omega_\phi, \mathbb{R})$ for all $\phi \in \mathcal{A}_0$, $R^\wedge \in \prod_{\phi\in\mathcal{A}_0} \mathcal{C}^\infty(\Omega_\phi, \mathbb{R})$. By Cartesian closedness of **Dlg**:

$$\boldsymbol{\mathcal{E}}^e(\Omega) \simeq \prod_{\phi\in\mathcal{A}_0} \mathcal{C}^\infty(\Omega_\phi, \mathbb{R}).$$

Therefore, up to smooth isomorphism, figures of $\boldsymbol{\mathcal{E}}^e(\Omega)$ can be described as maps $d : V \longrightarrow \prod_{\phi\in\mathcal{A}_0} \mathcal{C}^\infty(\Omega_\phi, \mathbb{R})$ such that $d(-)(\phi)^\vee \in \mathcal{C}^\infty(V \times \Omega_\phi, \mathbb{R})$ for every $\phi \in \mathcal{A}_0$.

The space $\mathcal{E}^e_M(\Omega)$

The natural diffeology on the space of moderate functions $\mathcal{E}^e_M(\Omega)$ is the sub-diffeology of $\boldsymbol{\mathcal{E}}^e(\Omega)$. Hence,

$$\boldsymbol{\mathcal{E}}^e_M(\Omega) := (\mathcal{E}^e_M(\Omega) \prec \boldsymbol{\mathcal{E}}^e(\Omega)) \in \mathbf{Dlg}.$$

Figures $d \in_V \boldsymbol{\mathcal{E}}^e_M(\Omega)$ are the maps $d : V \longrightarrow \mathcal{E}^e_M(\Omega)$ such that $d(-)(\phi, -) \in \mathcal{C}^\infty(V \times \Omega_\phi, \mathbb{R})$ for every $\phi \in \mathcal{A}_0$.

The special and full Colombeau algebras

Since the category **Dlg** of diffeological spaces is cocomplete, both quotient algebras $\mathcal{G}^s(\Omega)$ and $\mathcal{G}^e(\Omega)$ can be viewed as objects of **Dlg**:

$$\boldsymbol{\mathcal{G}}^s(\Omega) := \boldsymbol{\mathcal{E}}^s_M(\Omega)/\mathcal{N}^s(\Omega)$$
$$\boldsymbol{\mathcal{G}}^e(\Omega) := \boldsymbol{\mathcal{E}}^e_M(\Omega)/\mathcal{N}^e(\Omega).$$

Figures of these spaces can be described using the notion of quotient diffeology. For example, $d \in_U \boldsymbol{\mathcal{G}}^s(\Omega)$ iff $d : U \longrightarrow \mathcal{G}^s(\Omega)$ and for each $u \in U$, we can find an open neighborhood V of u in U and a map $\delta : V \longrightarrow \mathcal{C}^\infty(\Omega)^I$ such that

(i) $\delta(v)$ is moderate for each $v \in V$;
(ii) $(\delta^\vee)^\vee(-, \varepsilon, -) \in \mathcal{C}^\infty(V \times \Omega, \mathbb{R})$ for each $\varepsilon \in I$;
(iii) $d|_V = \pi \circ \delta$, where $\pi : (u_\varepsilon) \in \mathcal{E}^s_M(\Omega) \mapsto [u_\varepsilon] \in \mathcal{G}^s(\Omega)$ is the projection onto the quotient.

Analogously, we can describe figures of the full Colombeau algebra.

We can now state the following natural result:

Theorem 5.1 *Both for the special and the full Colombeau algebras $\mathcal{G}(\Omega) \in \{\boldsymbol{\mathcal{G}}^s(\Omega), \boldsymbol{\mathcal{G}}^e(\Omega)\}$, the sum, product and derivation maps*

$$+ : \mathcal{G}(\Omega) \times \mathcal{G}(\Omega) \longrightarrow \mathcal{G}(\Omega)$$
$$\cdot : \mathcal{G}(\Omega) \times \mathcal{G}(\Omega) \longrightarrow \mathcal{G}(\Omega)$$
$$\partial^\alpha : \mathcal{G}(\Omega) \longrightarrow \mathcal{G}(\Omega) \qquad \forall \alpha \in \mathbb{N}^n$$

are smooth. Therefore, with respect to the D-topology, $\mathcal{G}(\Omega)$ is a topological algebra with continuous derivations.

Moreover, let $(\psi_\varepsilon) \in \mathcal{D}(\Omega)^I$ *be a net verifying properties* (i), (ii), (iii), (iv), (v) *of Theorem* 1.1*, let* ι_Ω *be defined as in* (1.1)*, and let* $\sigma_\Omega(f) := [f] \in \mathcal{G}(\Omega)$ *for all* $f \in \mathcal{C}^\infty(\Omega)$*. Then, the embeddings*

$$\iota_\Omega : (|\mathcal{D}'(\Omega)| \prec \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})) \longrightarrow \mathcal{G}(\Omega)$$
$$\sigma_\Omega : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{G}(\Omega)$$

are smooth maps.

Proof We prove that the maps are smooth for the case $\mathcal{G}(\Omega) = \boldsymbol{\mathcal{G}}^s(\Omega)$, since the proof is similar for the case $\mathcal{G}(\Omega) = \boldsymbol{\mathcal{G}}^e(\Omega)$.

Concerning the smoothness of the sum map, let $d \in_U \boldsymbol{\mathcal{G}}^s(\Omega) \times \boldsymbol{\mathcal{G}}^s(\Omega)$, i.e., $p_i \circ d \in_U \boldsymbol{\mathcal{G}}^s(\Omega)$, where $p_i : \boldsymbol{\mathcal{G}}^s(\Omega) \times \boldsymbol{\mathcal{G}}^s(\Omega) \longrightarrow \boldsymbol{\mathcal{G}}^s(\Omega)$, $i = 1, 2$, are the projections. Hence, by the definition of the quotient diffeology on $\boldsymbol{\mathcal{G}}^s(\Omega)$, for any $u \in U$ we can write $(p_i \circ d)|_{V_i} = \pi \circ \delta_i$, where $V_i \in \tau_U$, $u \in V_1 \cap V_2$ and $\delta_i \in_{V_i} \boldsymbol{\mathcal{E}}^s_M(\Omega)$. Thus, we can write the composition

$$(+ \circ d)|_{V_1 \cap V_2} : v \mapsto \pi[\delta_1(v)] + \pi[\delta_2(v)] = \pi[\delta_1(v) + \delta_2(v)].$$

Since $(\delta_1 + \delta_2)|_{V_1 \cap V_2} \in_{V_1 \cap V_2} \boldsymbol{\mathcal{E}}^s_M(\Omega)$, the conclusion follows from the definition of the quotient diffeology.

Analogously, we can prove that the product map is smooth.

Concerning the smoothness of the partial derivative ∂^α, if $d \in_U \boldsymbol{\mathcal{G}}^s(\Omega)$, then for any $u \in U$ we can write $d|_V = \pi \circ \delta$, where $u \in V \in \tau_U$ and $\delta \in_V \boldsymbol{\mathcal{E}}^s_M(\Omega)$. Therefore, we have

$$(\partial^\alpha \circ d)|_V : v \mapsto \partial^\alpha(d(v)) = \partial^\alpha(\pi(\delta(v))) = \pi[\partial^\alpha \delta(v)].$$

And it is not difficult to show that $\partial^\alpha \in \mathcal{C}^\infty(\boldsymbol{\mathcal{E}}^s_M(\Omega), \boldsymbol{\mathcal{E}}^s_M(\Omega))$. Hence, $\partial^\alpha \delta \in_V \boldsymbol{\mathcal{E}}^s_M(\Omega)$, and the conclusion follows.

Concerning the smoothness of the embeddings, we only need to prove that ι_Ω is smooth, since the smoothness of σ_Ω follows directly from the definition of figures of a quotient diffeology. Let $d \in_U (|\mathcal{D}'(\Omega)| \prec \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R}))$, i.e., $d^\vee \in \mathcal{C}^\infty(U \times \mathcal{D}(\Omega), \mathbb{R})$. We can compute that

$$(\iota_\Omega \circ d)(u) = [d(u) * (\varepsilon \odot \psi_\varepsilon|_\Omega)]$$
$$= \left[x \in \Omega \mapsto \left\langle d(u), \frac{1}{\varepsilon^n} \psi_\varepsilon \left(\frac{x - \cdot}{\varepsilon} \right) \right\rangle \right].$$

For any fixed $\varepsilon \in I$, we show below that the map

$$\delta_\varepsilon : (u, x) \in U \times \Omega \mapsto \langle d(u), \frac{1}{\varepsilon^n} \psi_\varepsilon \left(\frac{x - \cdot}{\varepsilon} \right) \rangle = d^\vee \left[u, \frac{1}{\varepsilon^n} \psi_\varepsilon \left(\frac{x - \cdot}{\varepsilon} \right) \right] \in \mathbb{R}$$

is smooth. For the moderateness property, see [24,39]. Define the maps

$$S : (\varepsilon, \phi) \in I \times \mathcal{D}(\mathbb{R}^n) \mapsto \varepsilon \odot \phi \in \mathcal{D}(\mathbb{R}^n) \tag{5.1}$$
$$\widetilde{T} : (x, \phi) \in \mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n) \mapsto \phi(x - \cdot) \in \mathcal{D}(\mathbb{R}^n). \tag{5.2}$$

Then,

$$\delta_\varepsilon(u, x) = d^\vee[u, S(\varepsilon, \widetilde{T}(x, \psi_\varepsilon))] \quad \forall (u, x) \in U \times \Omega.$$

Therefore, δ_ε is smooth once we prove that both maps S and $\widetilde{T}$ are smooth. This is done in the following lemma:

Lemma 5.2 *The maps defined in* (5.1)*,* (5.2) *and*

$$T : (x, \phi) \in \mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n) \mapsto \phi(\cdot - x) \in \mathcal{D}(\mathbb{R}^n)$$

are smooth, i.e., $S \in \mathcal{C}^\infty(I \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n))$*,* $\widetilde{T} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n))$ *and* $T \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n))$*.*

Proof We only proceed for S, since the other two cases are similar. If $\varepsilon \in I$ and $p \in_U \mathcal{D}(\mathbb{R}^n)$, then $p^\vee \in \mathcal{C}^\infty(U \times \mathbb{R}^n, \mathbb{R})$ and $p^\vee$ is locally of uniformly bounded support with respect to U (Theorem 4.5). But $[S(\varepsilon, -) \circ p]^\vee(u, x) = \frac{1}{\varepsilon^n} p^\vee(u, \frac{x}{\varepsilon})$ for all $(u, x) \in U \times \Omega$, and this shows that $[S(\varepsilon, -) \circ p]^\vee \in \mathcal{C}^\infty(U \times \mathbb{R}^n, \mathbb{R})$ and it is locally of uniformly bounded support with respect to U.

Since I has the discrete diffeology, all these δ_ε's together induce a smooth map $\delta : U \longrightarrow \boldsymbol{\mathcal{E}}^s_M(\Omega)$ such that $\pi \circ \delta = \iota_\Omega \circ d$. By the definition of the quotient diffeology on $\boldsymbol{\mathcal{G}}^s(\Omega)$, the embedding $\iota_\Omega : (|\mathcal{D}'(\Omega)| \prec \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})) \longrightarrow \boldsymbol{\mathcal{G}}^s(\Omega)$ is smooth. □

5.1 Colombeau ring of generalized numbers and evaluation of generalized functions

In this subsection, we consider only the case of the special Colombeau algebra $\mathcal{G}^s(\Omega)$ since it is mostly studied in the literature. The case of the full algebra can be treated analogously.

One of the main features of Colombeau theory is the possibility to define a point evaluation for every CGF. Hence, it is natural to ask whether this evaluation map

$$\mathrm{ev} : (u, x) \in \boldsymbol{\mathcal{G}}^s(\Omega) \times \widetilde{\boldsymbol{\Omega}}_c \mapsto u(x) \in \widetilde{\boldsymbol{R}} \tag{5.3}$$

is smooth or not (see Sect. 1.1 for the definitions of $\widetilde{\Omega}_c$ and $\widetilde{\mathbb{R}}$), where the diffeologies we consider on $\widetilde{\Omega}_c$ and $\widetilde{\mathbb{R}}$ are the natural ones.

Definition 5.3 All the following are diffeological spaces:

(i) $\boldsymbol{R}_M := (\mathbb{R}_M \prec \mathcal{C}^\infty(I, \mathbb{R}))$;
(ii) $\widetilde{\boldsymbol{R}} := \boldsymbol{R}_M/\sim$;
(iii) $\boldsymbol{\Omega}_M := (\Omega_M \prec \mathcal{C}^\infty(I, \Omega))$;
(iv) $\widetilde{\boldsymbol{\Omega}} := \boldsymbol{\Omega}_M/\sim$;
(v) $\widetilde{\boldsymbol{\Omega}}_c := (\widetilde{\Omega}_c \prec \widetilde{\boldsymbol{\Omega}})$.

Note, e.g., that $d \in_U \boldsymbol{\Omega}_M$ iff $d : U \longrightarrow \Omega_M$ and $d^\vee(-, \varepsilon) \in \mathcal{C}^\infty(U, \Omega)$ for every $\varepsilon \in I$.

Theorem 5.4 *The evaluation map* (5.3) *is smooth.*

Proof Let $a \in_U \boldsymbol{\mathcal{G}}^s(\Omega)$ and let $b \in_U \widetilde{\boldsymbol{\Omega}}_c$. We need to prove that $\mathrm{ev} \circ (a, b) \in_U \widetilde{\boldsymbol{R}}$. For any fixed $u \in U$, by definition of the quotient diffeologies, we can write $a|_V = \pi_1 \circ \alpha$ and $i \circ b|_V = \pi_2 \circ \beta$, where $u \in V \in \tau_U$, $\alpha \in_V \boldsymbol{\mathcal{E}}^s_M(\Omega)$, $\beta \in_V \boldsymbol{\Omega}_M$, $i : \widetilde{\boldsymbol{\Omega}}_c \hookrightarrow \widetilde{\boldsymbol{\Omega}}$ is the inclusion, and $\pi_1 : \boldsymbol{\mathcal{E}}^s_M(\Omega) \longrightarrow \boldsymbol{\mathcal{G}}^s(\Omega)$, $\pi_2 : \boldsymbol{\Omega}_M \longrightarrow \widetilde{\boldsymbol{\Omega}}$ are the projections. Hence, for any $v \in V$, we have $(\mathrm{ev} \circ (a, b))(v) = \mathrm{ev}(a(v), b(v)) = \mathrm{ev}(\pi_1(\alpha(v)), \pi_2(\beta(v))) = \mathrm{ev}([(\alpha^\vee)^\vee(v, \varepsilon, -)], [\beta^\vee(v, \varepsilon)]) = [(\alpha^\vee)^\vee(v, \varepsilon, \beta^\vee(v, \varepsilon))]$. Note that for any $\varepsilon \in I$, $(\alpha^\vee)^\vee(-, \varepsilon, -) \in \mathcal{C}^\infty(V \times \Omega, \mathbb{R})$ and $\beta^\vee(-, \varepsilon) \in \mathcal{C}^\infty(V, \Omega)$, the restriction $(\mathrm{ev} \circ (a, b))|_V$ at ε can be written as an ordinary smooth function defined on V composed with the projection $\pi : \boldsymbol{R}_M \longrightarrow \widetilde{\boldsymbol{R}}$. Therefore, $\mathrm{ev} \circ (a, b) \in_U \widetilde{\boldsymbol{R}}$. □

6 Conclusions and open problems

We explore why the categories **Fr** of Frölicher spaces, **Dlg** of diffeological spaces and **FDlg** of functionally generated (diffeological) spaces work as good frameworks for the classical spaces of functional analysis and/or for Colombeau algebras, with some emphasis on **FDlg**. On the one hand, there seem to be few differences between **FDlg** and **Fr**. We can say that the former seems better than the latter because in **FDlg** we do not have the problem of extending locally defined functionals to the whole space; but it is easier to work directly with globally defined functionals when the D-topology of the space is unknown. On the other hand, when compared to diffeological spaces, functionally generated spaces seem to be closer to spaces used in functional analysis, where testing smoothness using functionals is customary. But Theorems 5.1 and 5.4 show that **Dlg** can be considered as a promising categorical framework for Colombeau algebras. Some open problems underscored by the present work are the following:

- A clear and useful example of functionally generated space which is not Frölicher, i.e., not every locally defined functional can be extended to the whole space, is missing.
- The problem to show that **Dlg** gives also a sufficiently simple infinite-dimensional calculus for the diffeomorphism invariant Colombeau algebra (see [24]) remains open. In particular, we note that the differentiable uniform boundedness principle [24, Thm. 2.2.7] is only used to prove the analogy of Lemma 5.2, whereas the other results of [24, Section 2.2.1] seem repeatable in **Dlg** without the need of knowing the calculus on convenient vector spaces.
- The relationship between the locally convex topology and the D-topology on $\mathcal{D}(\Omega)$ is only partially elucidated (see Corollary 4.12).
- The relationship between the space of Schwartz distributions $\mathcal{D}'(\Omega)$ and the smooth dual $\mathcal{D}(\Omega)'_s$ is only partially clarified (see Corollary 4.19).

- The problem of preservation of colimits from the category of smooth manifolds to **FDlg** is only partially solved.
- We do not know if **FDlg** is locally Cartesian closed, since the usual counter-examples about locally Cartesian closedness of **Fr** do not seem to work in **FDlg**.

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