

An extended standard model and its Higgs geometry from the matrix model

Harold C. Steinacker* and Jochen Zahn*

Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

*E-mail: harold.steinacker@univie.ac.at, jochen.zahn@univie.ac.at

Received May 9, 2014; Revised June 23, 2014; Accepted July 14, 2014; Published August 14, 2014

.....
We find a simple brane configuration in the IKKT matrix model which resembles the standard model at low energies, with a second Higgs doublet and right-handed neutrinos. The electroweak sector is realized geometrically in terms of two minimal fuzzy ellipsoids, which can be interpreted in terms of four point-branes in the extra dimensions. The electroweak Higgs connects these branes and is an indispensable part of the geometry. Fermionic would-be zero modes arise at the intersections with two larger branes, leading precisely to the correct chiral matter fields at low energy, along with right-handed neutrinos which can acquire a Majorana mass due to a Higgs singlet. The larger branes give rise to $SU(3)_c$, extended by $U(1)_B$ and another $U(1)$ which are anomalous at low energies and expected to disappear. At higher energies, mirror fermions and additional fields arise, completing the full $\mathcal{N} = 4$ supersymmetry. The brane configuration is a solution of the model, assuming a suitable effective potential and a non-linear stabilization of the singlet Higgs. The basic results can be carried over to $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills on ordinary Minkowski space with sufficiently large N .
.....

Subject Index B25, B41, B82, B83

1. Introduction

The main result of this paper is to establish a background of the IKKT or IIB model [1] with low-energy physics close to that of the standard model. This is part of the program of using matrix models as the basis for a theory of fundamental interactions and matter that has been pursued for many years from various points of view [2–11]. We focus here on the relation with particle physics, restricting ourselves to the case of flat four-dimensional space-time. Indeed, it is well known that flat Minkowski space arises as a “brane” solution of the IKKT model, realized as a non-commutative plane \mathbb{R}_θ^4 . It is also known that the fluctuations of the (bosonic and fermionic) matrices around a background consisting of N coincident such \mathbb{R}_θ^4 branes give rise to non-commutative maximally supersymmetric $\mathcal{N} = 4$ $U(N)$ super Yang–Mills (SYM) on \mathbb{R}_θ^4 ; see Refs. [4,8]. Accordingly, our results can be interpreted as statements within non-commutative $\mathcal{N} = 4$ $U(N)$ SYM, with sufficiently large N . In fact, most of the results also apply to $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills on ordinary Minkowski space, with sufficiently large N . The main difference lies in the $U(1)$ sector, which acquires a special role in the matrix model, related to the effective gravity [8,9]; however, we largely ignore this issue in the present paper.

At first sight, it may seem hopeless to obtain anything resembling the standard model from a maximally supersymmetric gauge theory. However, at low and intermediate energies this can be achieved. We establish certain backgrounds of the matrix model, interpreted as intersecting branes in six extra

dimensions, which lead to fermionic and bosonic low-energy excitations governed by an effective action which is close to the standard model, with all the correct quantum numbers. This is a very remarkable result, given the non-chiral nature of $\mathcal{N} = 4$ SYM. The price to pay are mirror fermions which arise at higher energies, along with Kaluza–Klein towers of massive fields, which ultimately complete the full $\mathcal{N} = 4$ spectrum. There is indeed no way to obtain the standard model without Higgs: If we switch off the Higgs, some of these mirror modes become (quasi-)massless, and combine with the standard model fermions to form non-chiral multiplets. In that respect the Higgs sector differs from the standard model: It arises from two doublets which are an intrinsic part of two minimal fuzzy spheres. The spontaneous symmetry breaking (SSB) pattern is thus more intricate than in the standard model, but this does not rule out the possibility that its fluctuations realize the physical Higgs. The remarkable point is that the separation into chiral standard-model fields and the mirror sector arises quite naturally on simple geometrical backgrounds, largely reproducing the essential features of the standard model at low energies.

Let us describe the brane configuration in some detail. Our background consists of a stack of three baryonic branes \mathcal{D}_B realized as fuzzy spheres (giving rise to $SU(3)_c \times U(1)_B$), a leptonic brane \mathcal{D}_l , and two other branes \mathcal{D}_u and \mathcal{D}_d . These branes are embedded in six extra dimensions, such that the standard model fermions arise at their intersections. The basic mechanism for obtaining chiral fermions on intersecting non-commutative branes was already found in [12]. However, in that work additional intersections led to unwanted fermions with the wrong chiralities, and the Higgs was missing. In the present paper, both problems are resolved, by realizing the Higgs as an intrinsic part of two minimal fuzzy ellipsoids (consisting of two quantum cells) which are part of \mathcal{D}_u and \mathcal{D}_d , respectively. These ellipsoids intersect \mathcal{D}_B and \mathcal{D}_l at their antipodal points, leading to localized chiral fermions. The electroweak $SU(2)_L$ gauge group arises as the two “left-handed” intersection loci on \mathcal{D}_u (resp. \mathcal{D}_d) coincide. This $SU(2)_L$ is broken by the Higgs, which is an intrinsic part of the branes. This provides a geometrical realization¹ of the electroweak symmetry breaking, which should also protect the Higgs mass to some extent from quantum corrections. An extra singlet Higgs S connecting \mathcal{D}_u with \mathcal{D}_l prevents a right-handed $SU(2)_R$, and breaks lepton number $U(1)_l$. It should also induce a Majorana mass term for ν_R .

At low energy, all the four-dimensional fermions arising on our background are massive Dirac fermions such as electrons² and massive quarks. Their left- and right-handed chiral components transform in different representations of the spontaneously broken (!) gauge group, coupling to the appropriate gauge bosons. For example, e_L and e_R arise on two different intersections of the branes, connected by the Higgs. The Higgs is, moreover, essential for the chiral nature of the fermions at the intersections.

We stress that our results and predictions for the fermionic would-be zero modes arising at the brane intersections are not only theoretical expectations, but can be verified numerically. In particular, we can compute the mass spectrum given by the eigenvalues of the internal Dirac operator \not{D}_{int} on our background, as well as the approximate localization and chirality of the corresponding fermionic modes. The results are consistent with the expectations. In particular, we clearly see near-zero modes which are localized as predicted on the intersecting branes, with the expected chiralities.

¹ The realization in terms of minimal fuzzy ellipsoids is in fact somewhat reminiscent of Connes’ interpretation of the Higgs connecting two “branes” [13].

² The neutrinos also arise with a right-handed partner.

Their eigenvalues approach zero for increasing N , with a clear gap to the next eigenvalues corresponding to mirror fermions. For a range of parameters we even find good quantitative agreement with our estimates for the Yukawa couplings, including the first series of mirror fermions.

Our brane configuration is a solution of the bare matrix model action, supplemented by a simple $SO(6)$ -invariant term in the potential. Although we add such a term by hand here (thus explicitly breaking supersymmetry), it seems plausible that (a more complicated form of) such a potential arises in the quantum effective action of the original model. This reflects the interaction of the branes extended in the extra dimensions, due to the conjectured—and to some extent verified [1,6,7,14–17]—relation with supergravity. The singlet Higgs S corresponds to an instability of the linearized wave operator, which we assume to be non-linearly stabilized.

Other ways to obtain chiral fermions in the IKKT model and similar models have been proposed in the literature. This includes warped extra dimensions [18], allowing the circumvention of the index theorem [19] which applies to product spaces $\mathbb{R}_\theta^4 \times \mathcal{K}$. However, no such warped solution of matrix models is known at present. Chiral fermions can be obtained in unitary matrix models [20], which in a sense have a built-in toroidal compactification. However, these models are not supersymmetric, which may lead to problems upon quantization. In string theory, there are many ways to obtain chiral fermions; however, this entails the vast landscape of string compactifications with its inherent lack of predictivity. Avoiding this is in fact one of the main motivations for the IKKT model. Nevertheless, many of the present ideas related to brane constructions of the standard model originate from string theory; see Refs. [21–28]. Finally, it seems likely that a somewhat adapted brane configuration can be found in the BFSS model [2,3].

We should also state the potential problems and pitfalls of our proposal. At some scale above the electroweak scale, mirror fermions come into play, which couple to the standard model gauge bosons, and may decay into standard model fermions via extra massive gauge bosons. In order to be at least near-realistic, there should be a sufficiently large gap between the electroweak scale and the scale of the mirror fermions. Unfortunately, at tree level, it turns out that this gap is not very large. However, we argue that quantum corrections should increase this gap, since a tower of massive Kaluza–Klein gauge bosons couples to the mirror fermions (as well as to the ordinary fermions) but not to the electroweak gauge bosons or the Higgs. Proton decay is prevented by baryon number conservation, which is violated only by a quantum anomaly.

The solution presented here a priori leads to two generations, which arise from two widely separated intersection regions of the underlying branes, with the same structure and chiralities. It seems straightforward to extend them to any even number of generations by introducing multiple branes. To get an odd number of generations is less clear; one possibility is that the singlet Higgs S leads to a deformation of the background and removes one intersection region.

At this point, it is perhaps a bit optimistic to hope that the backgrounds proposed here—with some adjustments—can be phenomenologically viable. On the other hand it seems at least conceivable, and the fact that we can get so close in this maximally (super)symmetric matrix model is certainly very remarkable. This should provide motivation to investigate these observations in more detail.

This paper is organized as follows. In Sect. 2, we collect the required facts about the matrix model, and recall the relation with non-commutative $\mathcal{N} = 4$ SYM. From that point on the paper may be read and interpreted by anyone familiar with $\mathcal{N} = 4$ SYM. In Sect. 3, the organization of the fermions and their quantum numbers is recalled from [12]. The central idea of the Higgs realized as an intrinsic part of a minimal brane is explained in Sect. 3.3. In Sect. 3.4 we give the brane solution (35) of the matrix model, which is the centerpiece of the paper. It is also spelled out with all branes in (97). The

rest of the paper is devoted to establishing the low-energy physics on this background. The chiral fermions at the brane intersections are established in Sect. 4 in the flat limit $N \rightarrow \infty$, where they become exactly massless. The case of finite N is then discussed in Sect. 4.3 using an ansatz motivated by the previous section, which allows the estimation of the corresponding Yukawa couplings. These are compared with numerical computations. The symmetry breaking and the resulting low-energy effective field theory is elaborated in Sect. 5, which allows us to make contact with the standard model. In the appendix, we elaborate the reduction of the fermions to four dimensions.

2. The matrix model

Our starting point is the IKKT or IIB model [1], which is given by the action

$$S_{\text{YM}} = \Lambda_0^4 \text{Tr}([X^A, X^B][X_A, X_B] + \bar{\Psi}\Gamma_A[X^A, \Psi]). \quad (1)$$

The indices A, B run from 0 to 9, and are raised or lowered with the invariant tensor η_{AB} of $SO(9, 1)$. The X^A are Hermitian matrices, i.e. operators acting on a separable Hilbert space \mathcal{H} , and Ψ is a matrix-valued Majorana–Weyl spinor of $SO(9, 1)$, with Clifford generators Γ_A . We also introduced a scale parameter Λ_0 with $\dim \Lambda_0 = L^{-1}$. This model enjoys the fundamental gauge symmetry

$$X^A \rightarrow U^{-1}X^AU, \quad \Psi \rightarrow U^{-1}\Psi U, \quad U \in U(\mathcal{H}), \quad (2)$$

as well as the ten-dimensional Poincaré symmetry

$$\begin{aligned} X^A &\rightarrow \Lambda(g)^A_B X^B, & \Psi_\alpha &\rightarrow \tilde{\pi}(g)_\alpha^\beta \Psi_\beta, & g &\in \widetilde{SO}(9, 1), \\ X^A &\rightarrow X^A + c^A \mathbb{1}, & c^A &\in \mathbb{R}^{10} \end{aligned} \quad (3)$$

and an $\mathcal{N} = 2$ matrix supersymmetry [1]. The tilde indicates the corresponding spin group. Defining the matrix Laplacian as

$$\square\Phi := [X_B, [X^B, \Phi]], \quad (4)$$

the equations of motion of the model take the form

$$\square X^A = [X_B, [X^B, X^A]] = 0 \quad (5)$$

for all A , assuming $\Psi = 0$.

2.1. Noncommutative branes and gauge theory

We focus on matrix configurations (in fact solutions, ultimately) which describe embedded non-commutative (NC) branes. This means that the X^A can be interpreted as quantized embedding functions [8]

$$X^A \sim x^A : \quad \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10} \quad (6)$$

of a $2n$ -dimensional manifold embedded in \mathbb{R}^{10} . More precisely, there should be a quantization map $\mathcal{Q} : \mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})$ which maps functions on \mathcal{M} to a non-commutative (matrix) algebra, such that commutators can be interpreted as quantized Poisson brackets, and \mathcal{A} as a quantized algebra of functions on \mathcal{M} . In the semi-classical limit indicated by \sim , matrices are identified with functions via \mathcal{Q} ; in particular, $X^A = \mathcal{Q}(x^A) \sim x^A$, and commutators are replaced by Poisson brackets. For a more extensive introduction see, e.g., Ref. [8]. Then the commutators

$$[X^A, X^B] \sim i\{x^A, x^B\} = i\theta^{\alpha\beta}(x)\partial_\alpha x^A \partial_\beta x^B \quad (7)$$

encode a quantized Poisson structure on $(\mathcal{M}^{2n}, \theta^{\alpha\beta})$. This Poisson structure sets a typical scale of non-commutativity Λ_{NC} . We will assume that $\theta^{\alpha\beta}$ is non-degenerate, so that the inverse matrix $\theta_{\alpha\beta}^{-1}$ defines a symplectic form on $\mathcal{M}^{2n} \subset \mathbb{R}^{10}$.

The prototype of such a non-commutative brane solution is given by the four-dimensional quantum plane \mathbb{R}_θ^4 , defined by $[\bar{X}^\mu, \bar{X}^\nu] = i\theta^{\mu\nu} \mathbb{1}$, where $\theta^{\mu\nu}$ has rank 4. It obviously satisfies $\square \bar{X}^A = 0$. We can assume that this plane is embedded along $\mu = 0, \dots, 3$, with $\bar{X}^a = 0$ for $a = 4, \dots, 9$. The (well-known) key observation is that fluctuations of the matrices around this background,

$$X^A = \bar{X}^A + \mathcal{A}^A, \quad (8)$$

describe (non-commutative) $\mathcal{N} = 4$ gauge theory on \mathbb{R}_θ^4 . Interpreting the fluctuations \mathcal{A}^A as $u(N)$ -valued functions in \mathbb{R}_θ^4 , the matrix model reduces to (cf. [4,8])

$$\begin{aligned} S_{\text{YM}} &= \frac{\Lambda_0^4}{(2\pi)^2} \int d^4x \sqrt{G} \operatorname{tr}_N \left(-\rho^{-1} (\mathcal{F}\mathcal{F})_G - 2G^{\mu\nu} D_\mu \mathcal{A}^a D_\nu \mathcal{A}_a + \rho [\mathcal{A}^a, \mathcal{A}^b] [\mathcal{A}_a, \mathcal{A}_b] \right. \\ &\quad \left. + \rho^{1/2} \bar{\Psi} (\tilde{\gamma}^\mu (i\partial_\mu + [\mathcal{A}_\mu, \cdot]) + \rho^{1/2} \Gamma^a [\mathcal{A}_a, \cdot]) \Psi \right) \\ &= \int d^4x \sqrt{G} \operatorname{tr}_N \left(-\frac{1}{4g_{\text{YM}}^2} (\mathcal{F}\mathcal{F})_G - \frac{1}{2} G^{\mu\nu} D_\mu \Phi^a D_\nu \Phi_a + \frac{1}{4} g_{\text{YM}}^2 [\Phi^a, \Phi^b] [\Phi_a, \Phi_b] \right. \\ &\quad \left. + \bar{\psi} \tilde{\gamma}^\mu (i\partial_\mu + [\mathcal{A}_\mu, \cdot]) \psi + g_{\text{YM}} \bar{\psi} \Gamma^a [\Phi_a, \psi] \right), \end{aligned} \quad (9)$$

where

$$\begin{aligned} X^\mu &= \bar{X}^\mu + \theta^{\mu\nu} \mathcal{A}_\nu, \quad \mu, \nu = 0, \dots, 3 \\ G^{\mu\nu} &= \rho \theta^{\mu\nu'} \theta^{\nu\nu'} \eta_{\mu'\nu'}, \\ \rho &= \sqrt{|\theta^{-1}|}, \\ \Phi^a &= \frac{\Lambda_0^2}{\pi} \mathcal{A}^a, \quad a = 4, \dots, 9 \\ \psi &= \frac{\Lambda_0^2}{2\pi} \rho^{1/4} \Psi, \\ \tilde{\gamma}^\mu &= \rho^{1/2} \theta^{\nu\mu} \gamma_\nu, \end{aligned} \quad (10)$$

and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$ is the $u(N)$ field strength. Since $|G| = |\eta| = 1$ in four dimensions, we will drop \sqrt{G} from now on. All fields take values in $u(N)$. In particular, we can read off the $u(N)$ coupling constant,

$$\frac{1}{4g_{\text{YM}}^2} = \frac{\Lambda_0^4}{(2\pi)^2} \rho^{-1}. \quad (11)$$

Although the action (9) is written in a way that looks like the standard $\mathcal{N} = 4$ SYM, it is in fact non-commutative $\mathcal{N} = 4$ SYM on \mathbb{R}_θ^4 . In the present paper, we will focus on those aspects where this distinction becomes (almost) irrelevant, emphasizing that the basic results also apply to standard $\mathcal{N} = 4$ SYM on commutative \mathbb{R}^4 .

To describe the internal or “extra-dimensional” sector described by the Φ^a or $X^{4\dots 9}$, we need to consider more general branes \mathcal{M}^{2n} . Being embedded in \mathbb{R}^{10} , they are equipped with the induced metric

$$g_{\alpha\beta}(x) = \partial_\alpha x^A \partial_\beta x_A, \quad (12)$$

which is the pull-back of η_{AB} . However, this is not the effective metric on \mathcal{M}^{2n} . It turns out that the effective action for fields and matter on such NC branes is governed by a universal effective metric

$G^{\alpha\beta}$ given by [8]:

$$G^{\alpha\beta} = \rho \theta^{\alpha\alpha'} \theta^{\beta\beta'} g_{\alpha'\beta'}, \quad \rho = \left(\frac{\det \theta_{\alpha\beta}^{-1}}{\det g_{\alpha\beta}} \right)^{\frac{1}{2(n-1)}} \quad (13)$$

for $n > 1$. This can be seen using the semi-classical form of the matrix Laplace operator³ [8]

$$\square \Phi = [X_A, [X^A, \Phi]] \sim -\rho^{-1} \square_G \phi \quad (14)$$

acting on scalar fields $\Phi \sim \phi$. Then the matrix equations of motion (5) take the simple form

$$0 = \square X^A \sim -\rho^{-1} \square_G x^A, \quad (15)$$

hence the embedding $x^A \sim X^A$ is given by harmonic functions on \mathcal{M} with respect to $G_{\alpha\beta}$.

The prime example of a compact non-commutative brane is the fuzzy sphere S_N^2 [30–32]. Its embedding in \mathbb{R}^3 is given in terms of three matrices $Y^i = cL^i$, where L^i is the generator of the N -dimensional irreducible representation of $\mathfrak{su}(2)$. Then

$$\begin{aligned} \square_Y Y^i &= 2c^2 Y^i, \\ \sum_{i=1}^3 (Y^i)^2 &= c^2 \frac{N^2 - 1}{4}. \end{aligned} \quad (16)$$

In this paper, we will give such a compactification in terms of stacks of suitable \mathcal{K} , resulting in a four-dimensional gauge theory on \mathbb{R}^4 that resembles the standard model at low energy.

The constructions of this paper also apply to $SU(N)$ $\mathcal{N} = 4$ SYM theory on ordinary \mathbb{R}^4 . Then the brane configurations become backgrounds of the six scalar fields, and our results state that the low-energy physics of such a background resembles that of the standard model.

3. The standard model from branes in the matrix model

3.1. Fields and symmetries

In order to recover the standard model from the matrix model, all fields must be realized as matrices in the adjoint of some big $U(N)$ gauge group. The $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group must arise at low energies from the fundamental $SU(N)$ gauge group by some symmetry breaking mechanism, and the standard model matter fields must transform in the appropriate way. It is quite remarkable that this is possible at all. Such an embedding of the standard model fields was given in [12] (cf. [33]), which we take as the starting point here. The fermionic matrices (including a right-handed neutrino) are realized as follows

$$\Psi = \begin{pmatrix} 0_2 & 0 & 0 & l_L & Q_L \\ & 0 & \begin{pmatrix} 0 & e_R \\ 0 & \nu_R \end{pmatrix} & Q_R \\ & & 0 & 0 \\ & & & 0_3 \end{pmatrix}, \quad (17)$$

where

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad Q_R = \begin{pmatrix} d_R \\ u_R \end{pmatrix}. \quad (18)$$

³ This result does not apply to the two-dimensional case, where a modified formula holds [29].

The electric charge Q and the weak hypercharge Y are then realized by the adjoint action of the following $SU(N)$ generators:

$$t_Q = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & -\frac{1}{3} \end{pmatrix}, \quad t_Y = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & -\frac{1}{3} \end{pmatrix}. \quad (19)$$

In particular, the Gell-Mann–Nishima relation

$$t_Q = t_3 + \frac{1}{2} t_Y, \quad t_3 = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & & 0 \end{pmatrix} \quad (20)$$

is satisfied. Furthermore, we need a mechanism which breaks the $U(N)$ gauge group down to the standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)$ (possibly extended by additional $U(1)$ factors), such that Q and Y arise as above. This can be achieved naturally by a suitable arrangement of stacks of compact branes in the extra dimensions, analogous to brane constructions in string theory [22–28]. In the matrix model, such a collection of coincident branes \mathcal{K}_i can be realized by block matrix configurations $X_{(i)}^a$ acting on $\mathcal{H}_{(i)} \cong \mathbb{C}^{N_i}$, cf. [34]. This suggests a brane configuration [12] with $2 + 1 + 1$ “electroweak” branes $2 \times \mathcal{D}_w \oplus \mathcal{D}_a \oplus \mathcal{D}_b$ leading to $SU(2)_L \times U(1)^3$, along with a “leptonic” brane \mathcal{D}_l which carries a $U(1)_l$ gauge group (corresponding to lepton number), and three coincident “baryonic” branes \mathcal{D}_B which carry the $SU(3)_c \times U(1)_B$ gauge group:

$$X_{(\text{naive})}^a = \begin{pmatrix} X_{(w)}^a \otimes \mathbb{1}_2 & & & & \\ & X_{(a)}^a & & & \\ & & X_{(b)}^a & & \\ & & & X_{(l)}^a & \\ & & & & X_{(B)}^a \otimes \mathbb{1}_3 \end{pmatrix}. \quad (21)$$

Here, k coincident branes are described by $(\dots) \otimes \mathbb{1}_k$. This background breaks⁴ the $U(N)$ gauge symmetry down to the product of $U(k_i)$ as follows:

$$U(N) \rightsquigarrow \text{diag}(U(2)_L, U(1), U(1), U(1), U(3)). \quad (22)$$

A priori, fermions on such non-commutative branes are not chiral, and thus cannot realize the standard model. Remarkably, chiral fermions do arise on intersections of such (non-commutative!) branes as shown in [12], provided they locally span the internal space \mathbb{R}^6 . Thus for suitable arrangements of the above branes, the required chiral fermions (17) may indeed arise on the corresponding intersections of \mathcal{D}_l and \mathcal{D}_B with the electroweak branes $\mathcal{D}_w, \mathcal{D}_a, \mathcal{D}_b$. However, due to the trivial topology of \mathbb{R}^{10} , there are always additional intersections, leading to additional fermions with the opposite

⁴ This is nothing but a variant of the usual Higgs mechanism, viewing the X^a as scalar fields. We assume that each $X_{(i)}^a$ generates the irreducible algebra $\text{Mat}(N_i, \mathbb{C})$ of functions on one brane \mathcal{D}_i .

chiralities. This is quite unavoidable for branes with product geometry⁵ $\mathbb{R}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$, as can be shown by an index theorem [19].

We propose a simple solution to this problem here, which at the same time provides a compelling mechanism for the electroweak Higgs. First, we note that the Higgs doublets

$$H_d = \begin{pmatrix} 0 \\ \phi_d \end{pmatrix}, \quad H_u = \begin{pmatrix} \phi_u \\ 0 \end{pmatrix}, \quad (23)$$

with $Y(H_d) = 1$ (as in the standard model) and $Y(H_u) = -1$ (as in the MSSM), fit into the above matrix structure as

$$X_{(H)}^a = \begin{pmatrix} 0_2 & H_d & H_u & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & S & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \phi_u & 0 & 0 \\ 0 & 0 & \phi_d & 0 & 0 & 0 \\ 0 & \phi_d^\dagger & 0 & 0 & 0 & 0 \\ \phi_u^\dagger & 0 & 0 & 0 & S & 0 \\ 0 & 0 & 0 & S^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

This indeed leads to the desired pattern of electroweak symmetry breaking. We also added a “sterile” Higgs S , which is a singlet under the standard model gauge group, occupying the same slot as ν_R . This leads to a modified matrix background of the form

$$X^a = X_{(\text{naive})}^a + X_{(H)}^a, \quad (25)$$

which, however, still does not resolve the problem of chirality doubling. The solution comes from replacing the two branes connected by the Higgs with a single non-commutative brane, recognizing the Higgs as an intrinsic part of the geometry. This is explained in the next section.

3.2. Higgs from deconstructing compact branes

For two branes connected by an off-diagonal Higgs as above, the embedding matrices generate an irreducible algebra which contains the original branes as sub-algebras, and should therefore be interpreted geometrically as a single compact space \mathcal{K} . Conversely, a single compact brane \mathcal{K} can be considered as a 2-brane system glued together at the boundary by some Higgs, as sketched in Figure 1. For example, S_N^2 can be interpreted as two disks in the xy direction near the north and south poles connected by an equatorial strip, which realizes the Higgs. In mathematical terms, we split the Hilbert space of a fuzzy sphere $\mathcal{H}_N = \mathbb{C}^N = \mathbb{C}^{N/2} \oplus \mathbb{C}^{N/2} \cong \mathcal{H}_{(1)} \oplus \mathcal{H}_{(2)}$ into two halves interpreted as $\mathcal{D}_{(1)}$ and $\mathcal{D}_{(2)}$, and write the embedding matrices as

$$X^a = \begin{pmatrix} X_{(1)}^a & 0 \\ 0 & X_{(2)}^a \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}. \quad (26)$$

We can then interpret the two diagonal blocks as two a priori separate branes, linked by the Higgs field ϕ . Note that $\mathcal{D}_{(1)}$ and $\mathcal{D}_{(2)}$ have opposite Poisson structure near the origin, and are transversally separated by the diameter. The two groups $\mathcal{U}_{(1)} = U(\mathcal{H}_{(1)})$ and $\mathcal{U}_{(2)} = U(\mathcal{H}_{(2)})$ corresponding to the diagonal blocks can be viewed as gauge groups on the two half-branes.⁶ Then ϕ intertwines these

⁵ Another possible solution to this problem was proposed in [18] based on a “warped” geometry. However, it is not clear how such configurations can arise in matrix models.

⁶ They should not be viewed as a stack of identical branes, because they have opposite orientation.

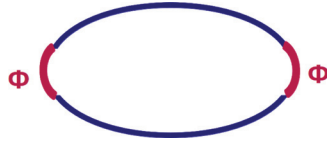


Fig. 1. Higgs from compactified branes.

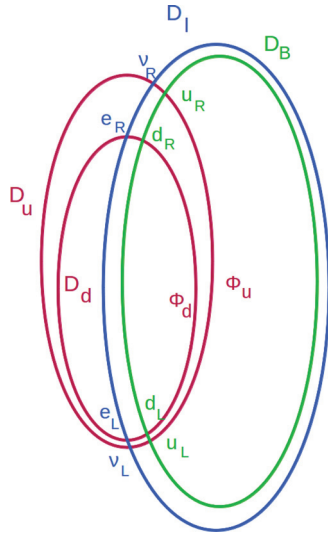


Fig. 2. SM brane scheme.

gauge groups, and plays the role of a Higgs. Indeed, the four-dimensional gauge fields corresponding to $\mathcal{U}_{(1)}, \mathcal{U}_{(2)}$ will acquire a mass due the Higgs effect.

One problem with this idea is that the Kaluza–Klein (KK) gauge modes on \mathcal{K} would not, in general, respect these upper or lower half-branes, but spread over the entire compact quantum space. Moreover, they would not respect the localized fermions arising on intersections of branes in a clear-cut way. These problems are avoided for fuzzy spaces with $N = 2$ represented on $\mathbb{C}^2 = \mathcal{H}_{(1)} \oplus \mathcal{H}_{(2)}$ with $\dim \mathcal{H}_{(1)}, \mathcal{H}_{(2)} = 1$; we call them minimal $N = 2$ quantum spaces. This leads to a simple set of gauge modes arising from a short KK tower.

3.3. Minimal electroweak branes with Higgs

Applying this idea to the above brane scheme, we interpret the Higgs ϕ_u as a fusion of (the “half-branes” defined by) the first and fourth lines in (24) into a single compact brane denoted by \mathcal{D}_u , and ϕ_d as a fusion of the second and third lines into another compact brane \mathcal{D}_d . If these two branes touch each other at some point, an approximate (i.e. spontaneously broken) $U(2)_L$ gauge group arises at the intersection, corresponding to the electroweak $SU(2)_L$ gauge group of the original stack of \mathcal{D}_w branes (22). If that common point of \mathcal{D}_u and \mathcal{D}_d is at the intersection with \mathcal{D}_l (and \mathcal{D}_B), then the chiral fermions arising at this location will transform as doublets under $SU(2)_L$. This leads to a brane scheme as sketched in Fig. 2. Although, e.g., \mathcal{D}_u also intersects \mathcal{D}_l at another point, leading to fermions with opposite chirality (as implied by the index theorem), these fermions now transform trivially under $SU(2)_L$. In this way, an effectively chiral theory can emerge from an underlying non-chiral model. $SU(2)_L$ is broken by the brane geometry, due to the Higgs identified above as an intrinsic part of the brane.

A simple explicit example of such a configuration is given by two fuzzy spheres embedded as follows:

$$X_4 = \begin{pmatrix} \phi_d \lambda_1 & \\ & \phi_u \lambda_1 \end{pmatrix}, \quad X_5 = \begin{pmatrix} \phi_d \lambda_2 & \\ & \phi_u \lambda_2 \end{pmatrix}, \quad X_6 = \begin{pmatrix} r_d \lambda_3 & \\ & r_u \lambda_3 + c \mathbb{1} \end{pmatrix}, \quad (27)$$

where λ_i are $\mathfrak{su}(2)$ generators in the N -dimensional representation. For r_u, r_d , and c appropriately chosen, they touch at the south pole p_- , which leads to an approximate (spontaneously broken) $SU(2)_L$ at p_- , as elaborated in Sect. 5. The corresponding Higgs ϕ will be identified shortly. These two fuzzy spheres realize the branes \mathcal{D}_u and \mathcal{D}_d touching each other. Then $\mathcal{D}_u \cap \mathcal{D}_B$ intersecting at p_- gives rise to u_L , and $\mathcal{D}_d \cap \mathcal{D}_B$ at p_- gives rise to d_L , such that (u_L, d_L) transform as a doublet under $SU(2)_L$. Similarly, $\mathcal{D}_u \cap \mathcal{D}_B$ intersecting at the north pole gives u_R , and $\mathcal{D}_d \cap \mathcal{D}_B$ gives d_R . These do not transform under $SU(2)_L$. In the same way, $\mathcal{D}_d \cap \mathcal{D}_l$ and $\mathcal{D}_u \cap \mathcal{D}_l$ intersecting at p_- gives rise to e_L and ν_L , while ν_R and e_R arise at their north poles. Again, (ν_L, e_L) form a doublet under $SU(2)_L$, while ν_R and e_R transform trivially.

In general, the physically relevant four-dimensional gauge fields are determined by a Kaluza–Klein mode expansion on \mathcal{D}_u and \mathcal{D}_d . These modes will in general not respect the decomposition into upper and lower halves, and couple to fermions with both chiralities to some extent. This problem is resolved if these two fuzzy spheres are realized by $S^2_{N=2}$, with minimal Hilbert spaces $\mathcal{H} \cong \mathbb{C}^2$. This leads to the following “minimal” electroweak matrix configuration:

$$X_4 \pm iX_5 = \frac{1}{2} \begin{pmatrix} \phi_d \sigma^\pm & \\ & \phi_u \sigma^\pm \end{pmatrix}, \quad (28)$$

$$X_6 = \frac{1}{2} \begin{pmatrix} r_d \sigma_3 & \\ & r_u \sigma_3 + c \mathbb{1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} r_d & & \\ & -r_d & \\ & & r_u + c \\ & & & -r_u + c \end{pmatrix}, \quad (29)$$

visualized in Fig. 3. Note that X^6 has four eigenvalues, two of which coincide if $c = r_u - r_d$. In the absence of ϕ , the unbroken gauge group given by the commutant (stabilizer) of this background is therefore $U(2)_L \times U(1) \times U(1)$ in that case.⁷ The gauge modes which do not commute with X^6 acquire a mass $m^2 \sim [X_6, [X_6, \cdot]]$ given by the difference of the X^6 eigenvalues. The $U(2)_L$ is broken in the presence of ϕ , which will play the role of the electroweak Higgs. Furthermore, it turns out that chiral fermionic zero modes arise at the intersections even for this very fuzzy geometry, realized by coherent states $|\pm\rangle_{u,d}$ on the branes located at the poles p_\pm . This will be verified explicitly in Sect. 4. These fermions couple to the low-energy gauge group as required, and are connected by the $\phi_{u,d}$. Although these $\phi_{u,d}$ clearly correspond to the electroweak Higgs sector, the precise role of their fluctuations and the relevance of the other geometrical moduli in the above configuration remains to be clarified.

It turns out that in order to have a configuration which is a solution to our modified action, we have to take $r_d = r_u$ and $c = 0$. Then one also has a broken $U(2)_R$ at the north pole. This will be broken not only by the above Higgs ϕ , but also at a higher energy scale by a non-vanishing expectation value of a singlet Higgs S . It connects \mathcal{D}_u and \mathcal{D}_l at the north pole of \mathcal{D}_u , thus lifting the degeneracy of the north poles of \mathcal{D}_u and \mathcal{D}_d . This is elaborated below. In particular, the breaking of the right-handed $SU(2)_R$ is discussed in Sect. 5.

⁷ The fate of the various $U(1)$ factors will be discussed in Sect. 5.

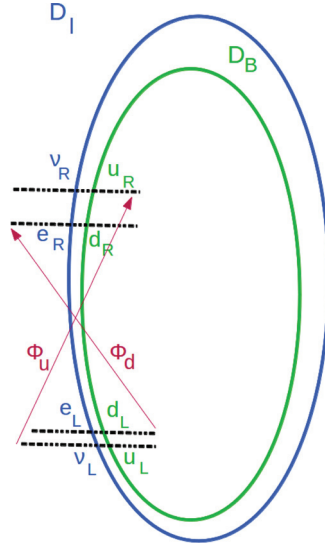


Fig. 3. Minimal SM brane scheme.

3.3.1. Discussion. Before establishing these claims in more detail, we briefly discuss some of the further issues arising in this scenario.

We need to specify the dimensions and type of the various branes. First, the above remark suggests that all four electroweak D0 branes corresponding to $|\pm\rangle_{u,d}$ should be located on both branes \mathcal{D}_B and \mathcal{D}_I , in order to obtain chiral near-zero modes which couple appropriately to the electroweak gauge fields. This suggests that \mathcal{D}_B and \mathcal{D}_I should be (nearly) coincident.

To get chiral fermions, the branes must span the internal space \mathbb{R}^6 at the intersections. Thus we have two possibilities: either the electroweak branes $\mathcal{D}_{u,d}$ are two dimensional and $\mathcal{D}_{l,B}$ are four dimensional, or conversely. This choice affects the effective four-dimensional gauge couplings, via the volume or trace over the extra dimensions. It turns out that the first possibility leads to a pattern of the electroweak gauge coupling constants (in particular the Weinberg angle) which seems unrealistic. We therefore take $\mathcal{D}_{l,B}$ to be two-dimensional fuzzy branes K_{N_2} with large N_2 , while the $\mathcal{D}_{u,d}$ have the structure $K_{N_1} \times S^2_{N=2}$. The extra K_{N_1} does not significantly change the above picture of the electroweak symmetry breaking, and merely introduces a multiplicative factor to the low-energy gauge groups. This allows a reasonable pattern of low-energy coupling constants, as discussed in Sect. 5.

An important question is the fate of the extra $U(1)$ gauge fields, which always arise in similar brane constructions [22–28]. Each brane comes with an associated $U(1)$ acting with $\mathbb{1}_i$ on $\mathcal{H}_{(i)}$, which do not acquire any mass term from a Higgs mechanism. The trace- $U(1)$ decouples completely in the commutative limit (i.e. for ordinary $\mathcal{N} = 4$ SYM), and can be identified with a gravitational mode on non-commutative space-time [8,9]; we will therefore ignore it in the present paper. Furthermore, a $U(1)_B \sim \frac{1}{3}\mathbb{1}_B$ corresponding to baryon number B arises on the baryonic brane \mathcal{D}_B , and a $U(1)_l \sim \mathbb{1}_l$ corresponding to lepton number l arises on the leptonic brane \mathcal{D}_l . Some of these will be affected by quantum anomalies, as discussed later. Most importantly, the electric charge

$$t_Q = \frac{1}{2} \left(\mathbb{1}_u - \mathbb{1}_d + \mathbb{1}_l - \frac{1}{3}\mathbb{1}_B \right) \quad (30)$$

also arises in this way, which is of course anomaly free.

Finally, quantum effects are expected to play an important role. Besides introducing (benign) anomalies, they will also mediate the interaction between the branes, which is expected to play an essential role in selecting and stabilizing the appropriate brane configuration. This should be a central theme for future work in this context.

3.3.2. Singlet Higgs S . To avoid an exactly massless $U(1)_{B-L}$ gauge field and to break $SU(2)_R$ for $r_u = r_d$, we assume that there is an extra singlet Higgs S connecting \mathcal{D}_u with \mathcal{D}_l at the location of ν_R . S can be seen as superpartner of ν_R , and it is a singlet of the standard model gauge group [cf. (19)]. In the presence of a VEV $\langle S \rangle \neq 0$, the \mathcal{D}_l and \mathcal{D}_u branes are unified into a single compact brane, which is natural in view of $t_Q = \frac{1}{2}(\mathbb{1}_u + \mathbb{1}_l - \dots)$. Clearly $\langle S \rangle$ breaks $U(1)_{B-L}$, leaving only one extra $U(1)_5$ gauge field besides the standard model gauge group at low energies. That $U(1)_5$ acquires a mass by the electroweak Higgs, and is anomalous at low energies. Such anomalous $U(1)$ gauge fields are expected to disappear from the low-energy spectrum by some variant of the Stückelberg mechanism, as discussed, e.g., in [35–37]. The symmetry breaking will be discussed in more detail in Sect. 5.

Finally, $\langle S \rangle \neq 0$ allows us to write down a Majorana mass term for the right-handed neutrino, such as

$$\int d^4x \operatorname{tr}_N (\nu_R^T \gamma^0 S^\dagger \nu_R S^\dagger). \quad (31)$$

Such a term is compatible with the full $SU(N)$ gauge symmetry, and could therefore arise in the quantum effective action even at high scales.

3.4. Intersecting brane solutions

In general, compact quantum spaces in Euclidean signature are never solutions of the classical matrix equation of motion $\square X^a = 0$. However, quantum effects will contribute to the effective action. It is generally expected that this can be related to some sort of effective (super-)gravity in higher dimensions; for some partial results from the matrix model point of view see, e.g., [1, 6, 7, 14–17]. In particular, this should lead to an attractive interaction between nearly coincident branes, and it is plausible that suitable compact brane configurations may be stabilized in this way. Lacking more specific results, we will model these quantum contributions to the effective potential on a four-dimensional space-time \mathbb{R}_θ^4 by an $SO(6)$ -invariant function $f(\operatorname{tr}_N \sum_{a=4}^9 X_a X^a)$:

$$S_{\text{YM}} \rightarrow S_{\text{YM}} - \rho \int d^4x \sqrt{G} V_{\text{quant}}, \quad (32)$$

$$V_{\text{quant}} = f \left(\Lambda_0^2 \operatorname{tr}_N \sum_{a=4}^9 X_a X^a \right) = f \left(\pi \rho^{-\frac{1}{2}} g_{\text{YM}} \operatorname{tr}_N \sum_{a=4}^9 \Phi_a \Phi^a \right). \quad (33)$$

Note that the non-commutativity scale ρ makes it possible to write down a dimensionless invariant radius operator. This leads to the equations of motion⁸

$$\square_X X^a = -2\pi g_{\text{YM}} \rho^{-\frac{1}{2}} f' X^a, \quad (34)$$

which will have non-trivial brane solutions (reflecting the above discussion) provided $f' < 0$ in some range. In particular, we give a solution with the properties discussed above, where \mathcal{D}_l and \mathcal{D}_B are

⁸ Note that the regularization for the matrix model proposed in [10, 11] also leads to the same type of equations of motion.

realized as fuzzy spheres $S_{N_l}^2$ and a stack of three $S_{N_B}^2$, while the electroweak branes \mathcal{D}_u and \mathcal{D}_d are realized as $S_{N_u}^2 \times S_{N=2}^2$ and $S_{N_d}^2 \times S_{N=2}^2$.

Recall that a fuzzy sphere S_N^2 is the matrix algebra $\text{Mat}(N, \mathbb{C})$ generated by the spin $\frac{N-1}{2}$ representation of $\mathfrak{su}(2)$,

$$[L_i, L_j] = i\varepsilon_{ijk}L_k,$$

with radius $\frac{1}{2}\sqrt{N^2 - 1} \sim N/2$. Denote the generators of $S_{N_u}^2$ by L_i and those of $S_{N_l}^2$ by K_i . The generators of S_2^2 are σ'_i , which are the Pauli matrices σ_i divided by 2. We also use the notation $L_{\pm} = L_1 \pm iL_2$. Now let $\mathcal{D}_u = S_{N_u}^2 \times S_2^2$ and $\mathcal{D}_l = S_{N_l}^2$ be embedded as

$$X_{(u)}^a = \begin{pmatrix} R'_u L_3 + \phi_u \sigma'_1 \mathbb{1}_{N_u} \\ \phi_u \sigma'_2 \mathbb{1}_{N_u} \\ r_u \sigma'_3 \mathbb{1}_{N_u} \\ 0 \\ R_u L_1 \\ R_u L_2 \end{pmatrix}, \quad X_{(l)}^a = \begin{pmatrix} R'_l K_3 \\ 0 \\ R_l K_1 \\ R_l K_2 \\ 0 \\ 0 \end{pmatrix}, \quad (35)$$

and analogously for \mathcal{D}_d and \mathcal{D}_B . This defines the basic background solutions under consideration here. The equations of motion (34) are satisfied provided

$$2R_u^2 = r_u^2 + \phi_u^2 = 2R_l^2 = R_l^2 + R_l'^2 = 2\phi_u^2 = R_u'^2 + R_u^2 = -2\pi g_{\text{YM}} \rho^{-\frac{1}{2}} f', \quad (36)$$

which implies

$$R_u^2 = R_u'^2 = R_l^2 = R_l'^2 = r_u^2 = \phi_u^2 = -\pi g_{\text{YM}} \rho^{-\frac{1}{2}} f', \quad (37)$$

and similarly for the other branes. Nevertheless, since the above effective action is certainly oversimplified, we will keep the different geometrical moduli r_i , ϕ_i , R_i henceforth, assuming only that they have the same scale.

The above brane configuration can alternatively be obtained as the solution of the bare $\mathcal{N} = 4$ SYM equations of motion on \mathbb{R}^4 without quantum corrections, by letting them rotate in the extra dimensions as $X^a = R_b^a(t) \bar{X}^b$, where \bar{X}^a is given by (35); cf. [38,39]. This is indeed a solution for suitable rotations in the 4–5, 6–7, and 8–9 planes. However, the rotation may distort the low-energy effective field theory in four dimensions, and we will not pursue this possibility here.

3.4.1. Intersections. Now consider the intersections of these branes. If $r, \phi \ll RN_{u,l}$, then \mathcal{D}_u and \mathcal{D}_l intersect near $L_{\pm} \approx 0 \approx K_{\pm}$, provided $R'_u L_3 \approx R'_l K_3$ up to corrections of order ϕ . This requires $R'_2 N_l \approx R'_u N_u$. There are hence two widely separated intersection regions located in target space approximately at $\pm R'_2 N_l (1, 0, 0, 0, 0, 0)$. Since the spheres are oriented, the helicity of the would-be zero modes is the same in the two intersection regions, as discussed in Sect. 4. These two intersection regions could therefore be interpreted in terms of two generations. Alternatively, a deformation of \mathcal{D}_l (e.g. by the singlet Higgs S) might remove one of these intersection regions, leaving only one generation at this stage.⁹

For the intersections of the other brane pairs, we similarly need $R'_2 N_l \approx R'_d N_d \approx R'_2 N_B$. We will also impose $N_u = N_d$, so that $SU(2)_L$ can act naturally on the Hilbert spaces of \mathcal{H}_u and \mathcal{H}_d , as

⁹ For example, this is achieved by slightly shifting one end of \mathcal{D}_l .

discussed below. To satisfy all these conditions,¹⁰ it follows that $N_l \approx N_B$, and $R'_u \approx R'_d$, so that $\mathcal{D}_u \approx \mathcal{D}_d$ and $\mathcal{D}_l \approx \mathcal{D}_B$ from a geometric point of view.

3.5. Flat limit $S_N^2 \rightarrow \mathbb{R}_\theta^2$

To understand the intersections discussed above for small but finite r, ϕ in a simple way, we want to approximate the large fuzzy spheres near these intersections by tangential quantum planes. We will thus replace \mathcal{D}_u by $\mathbb{R}_{89}^2 \times S_2^2$ and \mathcal{D}_l by \mathbb{R}_{67}^2 . In the limit of large N , the tangent space to a “point” on the fuzzy sphere generated by RL_i tends indeed to the quantum plane \mathbb{R}_θ^2 , if accompanied by a suitable scaling of R . As the number of Planck cells is N and the area is proportional to $R^2 N^2$, R should scale as $R \sim N^{-\frac{1}{2}}$ in order to have a constant Planck cell size and thus a well-defined flat limit. However, in the above configuration, a scaling of $R_{u,l}^{(i)}$ would have to be accompanied by a scaling of ϕ, r . Hence, in order to keep ϕ, r constant, we keep $R_{u,l}^{(i)}$ constant, and thus obtain a quantum plane with non-commutativity $\theta \sim N$. Specifically, we can replace the tangent spaces of the large spheres by quantum planes

$$L_\pm \rightarrow \sqrt{N_u}(y^8 \pm iy^9), \quad L_3 \rightarrow \pm \frac{N_u}{2} \left(1 - \frac{2}{N_u^2}(y_8^2 + y_9^2) + \mathcal{O}\left(\frac{y}{N}\right)^4 \right), \quad (38)$$

$$K_\pm \rightarrow \sqrt{N_l}(y^6 \pm iy^7), \quad K_3 \rightarrow \pm \frac{N_l}{2} \left(1 - \frac{2}{N_l^2}(y_6^2 + y_7^2) + \mathcal{O}\left(\frac{y}{N}\right)^4 \right), \quad (39)$$

embedded in the 8–9 and the 6–7 directions, where the y^i fulfill standard commutation relations

$$[y^6, y^7] = \pm i, \quad [y^8, y^9] = \pm i.$$

Here the sign depends on the sign of L_3 and K_3 respectively, and determines the chiralities of the would-be zero modes. Then the effective non-commutativity of the tangential generators $x^a = R\sqrt{N}y^a$ is given by $\theta \sim R^2 N$, and, e.g., the equation of motion (37) for the 8–9 components becomes

$$\frac{1}{N}\theta_{89} \left(1 + \frac{(R'_u)^2}{R_u^2} \right) = -2\pi g_{\text{YM}} f' \rho^{-\frac{1}{2}}. \quad (40)$$

We can now describe the intersections of \mathcal{D}_u with \mathcal{D}_l in more detail, assuming $r, \phi \ll NR$. In the limit of large N , we can write

$$X_{(u)}^a = \begin{pmatrix} \pm \frac{1}{2} R'_u N_u + \phi \sigma'_1 \\ \phi \sigma'_2 \\ r \sigma'_3 \\ 0 \\ R_u \sqrt{N_u} y^8 \\ R_u \sqrt{N_u} y^9 \end{pmatrix}, \quad X_{(l)}^a = \begin{pmatrix} \pm \frac{1}{2} R'_l N_l \\ 0 \\ R_l \sqrt{N_l} y^6 \\ R_l \sqrt{N_l} y^7 \\ 0 \\ 0 \end{pmatrix}. \quad (41)$$

Assuming $R'_u N_u = R'_l N_l$ to have perpendicular intersections, this reduces to the intersections of a minimal ellipsoid with a quantum plane, $\mathbb{R}_{\theta(89)}^2 \times S_2^2 \cap \mathbb{R}_{\theta(67)}^2$. The picture of intersecting branes

¹⁰ As discussed later, one way to introduce additional generations is via additional branes $\mathcal{D}_{u,d}^{(i)}$. Their parameters $R^{(i)}$ and their quantum numbers $N^{(i)}$ should be very close to each other to ensure that they all intersect with the same $\mathcal{D}_{l,B}$ branes. This leads to further constraints, and different $N^{(i)}$ are possible only if $R'_{u,d} \leq \phi$.

makes sense even for minimal fuzzy spheres $S_{N=2}^2$, since their coherent states are located at the corresponding classical ellipsoid

$$\frac{x_4^2 + x_5^2}{\phi^2} + \frac{x_6^2}{r^2} = 1. \quad (42)$$

Taking into account the curvature of \mathcal{D}_l near $y = 0$, the intersection is determined by the 456 coordinates

$$\begin{pmatrix} N_u R'_u + \phi \sin \varphi \\ 0 \\ r \cos \varphi \end{pmatrix} = \begin{pmatrix} N_l R'_l \cos \vartheta \\ 0 \\ N_l R_l \sin \vartheta \end{pmatrix}, \quad (43)$$

where φ is the angle on the normalized minimal fuzzy sphere, and ϑ on the large circle of \mathcal{D}_l . This suggests the following ansatz for the would-be zero modes,

$$\psi = |\varphi+\rangle_u \langle \vartheta|_l \otimes |s_{\underline{l}}\rangle, \quad (44)$$

in terms of coherent states located at their classical intersection; here, $|\varphi+\rangle_u$ is the product of coherent states¹¹ located at the angle φ on S_2^2 and at the north pole $L_3 = +\frac{N_u-1}{2}$ of $S_{N_u}^2$, $\langle \vartheta|_l$ is a coherent state on $S_{N_l}^2$ located at the angle ϑ , and $|s_{\underline{l}}\rangle$ indicates a suitable spinor state. It is not hard to see that this leads to approximate zero modes, consistent with the picture expected from the flat limit. However, we largely restrict ourselves to the flat limit in this paper, as elaborated below.

3.6. The singlet Higgs S

To complete the background, we have to discuss the singlet Higgs S , linking \mathcal{D}_u with \mathcal{D}_l at ν_R . Such a link between two branes will be localized at one (or both) of their intersections, suggesting an ansatz

$$H_{(S)}^a = h^a S + h.c., \quad S = \sum_n |p_n+\rangle_u \langle q_n|_l. \quad (45)$$

Here, $|p+\rangle_u$ denotes the tensor product of a state p on S_2^2 and the coherent state $L_3|+\rangle = \frac{N_u-1}{2}|+\rangle$ on $S_{N_u}^2$, and q is a state on $S_{N_l}^2$. For h , we choose the ansatz

$$h^a = h(e^8 + ie^9), \quad (46)$$

so that

$$X^a h_a = h R_u L_+. \quad (47)$$

Now we study the linearized wave operator on the perturbation H^a , which reads

$$(PH)^a = [X^b, [X_b, H^a]] + 2[[X^a, X^b], H_b] - [X^a, [X^b, H_b]] + 2\pi g_{\text{YM}} \rho^{-\frac{1}{2}} f' H^a. \quad (48)$$

Due to (47), we have

$$[X^b, h_b S] = h R_u e^{i\omega s t} L_+ \sum_n |p_n+\rangle_u \langle q_n|_l = 0.$$

Similarly, we compute, for $a \in \{4, 5, 6, 7\}$,

$$[[X^a, X^b], h_b S] = R_u R'_u h e^{i\omega s t} \delta_4^a L_+ \sum_n |p_n+\rangle_u \langle q_n|_l = 0.$$

¹¹ Coherent states on fuzzy spheres are obtained by $SO(3)$ rotations of the highest weight states; see [41,42].

It follows that, for $a \notin \{8, 9\}$, the equation of motion (48) is fulfilled. For $a \in \{8, 9\}$, we note that

$$[[X^a, X^b], h_b S] = -\frac{1}{2} R_u^2 (N_u - 1) h^a S,$$

$$\sum_{b=8}^9 [X^b, [X^b, h^a S]] = \frac{1}{2} R_u^2 (N_u - 1) h^a S.$$

We choose the state $\sum_n p_n \otimes q_n$ such that

$$\sum_{b=4}^7 [X^b, [X^b, S]] = \lambda S. \quad (49)$$

The double commutator on the left-hand side is a Hermitian operator on $\mathbb{C}^2 \otimes \mathbb{C}^{N_l}$, which can be diagonalized. One would expect that the two lowest eigenstates λ are localized near the intersections $(0, 0, \pm 1)$ on S_2^2 and $S_{N_l}^2$, suggesting the ansatz $S = |\varphi+\rangle_u \langle \vartheta|_l$. Then one expects

$$\lambda \approx \frac{1}{2} \phi^2 + \frac{1}{2} R_l^2 (N_l - 1) \quad (50)$$

for large N . This result, and also the localization, can be checked numerically with very high accuracy. Choosing one of these corresponds to either coupling the right- or the left-handed neutrino to the scalar Higgs. For the action of the wave operator (48) on (45), we thus obtain

$$(PH)^a = \left(-\frac{1}{2} R_u^2 (N_u - 1) + 2\pi \rho^{-\frac{1}{2}} g_{\text{YM}} f' + \lambda \right) H^a$$

$$\approx \left(\frac{1}{2} R_u^2 (N_l - N_u) - \frac{3}{2} \phi^2 \right) H^a,$$

where we used (37). In order to have the correct intersection, we need $N_l = N_u$. Hence, our ansatz (45) corresponds to a negative mode of the linearized wave operator, i.e., an instability. In the following, we assume that it is non-linearly stabilized, so that h acquires a non-trivial value. We plan to address this issue in a forthcoming paper.

The seemingly ad hoc coupling of S to \mathcal{D}_u rather than \mathcal{D}_d can be interpreted as spontaneous breaking of the $SU(2)_L \times SU(2)_R$ gauge symmetry down to $SU(2)_L \times U(1)$. This is discussed further in Sect. 5. Furthermore, the back-reaction of S to $\mathcal{D}_{u,l}$ might lead to a shift of the branes, possibly removing the other intersection regions of the branes (e.g. at $X^4 \approx -RN$). Then a single generation would arise for the above background.

4. Chiral fermions in the flat limit $N \rightarrow \infty$

In this section, we consider the limit $N \rightarrow \infty$, where the fuzzy spheres S_N^2 become much larger than the minimal electroweak branes $S_{N=2}^2$. We can then replace S_N^2 by a quantum plane \mathbb{R}_θ^2 near the intersection with S_2^2 , and obtain exact results for the (would-be) chiral fermions. This allows us to understand the resulting low-energy physics in a simple way.

4.1. S^2 intersecting \mathbb{R}^2

We want to understand the origin of massless chiral fermions arising on the intersection of the above minimal ellipsoids embedded in the 456 directions with a flat brane \mathbb{R}_θ^2 in the 6–7 plane, dropping the 89 directions for now. Thus, consider \mathcal{D}_u realized by $\text{Mat}(2, \mathbb{C})$ acting on $\mathcal{H}_{(u)} \cong \mathbb{C}^2$, and \mathcal{D}_l

realized by an operator algebra acting on $\mathcal{H}_{(l)}$. We should therefore find the (near-)zero modes of the “internal” Dirac operator in the 4567 direction:

$$\mathcal{D}_{\text{int}} \Psi = \sum_{a=4}^7 \Delta_a [X_a, \Psi] = \sum_{a=4}^7 \Delta_a (X_{(u)}^a \Psi - \Psi X_{(l)}^a) \quad (51)$$

(using the conventions of Appendix A) for the off-diagonal fermions $\Psi \in \mathcal{H}_{(u)} \otimes \mathcal{H}_{(l)}^*$ for a background of two branes

$$X^a = \begin{pmatrix} X_{(u)}^a & \\ & X_{(l)}^a \end{pmatrix}. \quad (52)$$

Here, Δ_a are the $SO(6)$ Gamma matrices.

As a warm-up, consider first the intersection of a single $D0$ brane with \mathbb{R}_θ^2 (see [43]). The $D0$ brane is given by a projector $X_{(u)}^a = p^a |p\rangle_u \langle p|_u$ located at $p^a = (p^4, p^5, p^6, 0)$. Then the fermions linking the state $|p\rangle_u$ with \mathbb{R}_θ^2 have the form

$$\Psi_p = |p\rangle_u \langle \psi|_l \quad (53)$$

for some state ψ on \mathbb{R}_θ^2 . Then we can write

$$\begin{aligned} \mathcal{D}_{\text{int}} \Psi &= \sum_{a=4,5,6} \Delta_a p^a \Psi - \sum_{a=6,7} \Delta_a \Psi X_{(l)}^a \\ &= \sum_{a=4,5} \Delta_a p^a \Psi - \left(\Delta_6 \Psi (X_{(l)}^6 - p_6) + \Delta_7 \Psi X_{(l)}^7 \right) \\ &=: \mathcal{D}_{(1)} \Psi - \mathcal{D}_{(2)} \Psi, \end{aligned} \quad (54)$$

so that $\{\mathcal{D}_{(1)}, \mathcal{D}_{(2)}\} = 0$, and $\mathcal{D}_{\text{int}}^2 = \mathcal{D}_{(1)}^2 + \mathcal{D}_{(2)}^2$. Therefore Ψ is a zero mode if and only if $\mathcal{D}_{(1)} \Psi = 0 = \mathcal{D}_{(2)} \Psi$. Clearly $\mathcal{D}_{(1)} \Psi = 0$ if and only if $p_4 = p_5 = 0$, i.e. p is located in \mathcal{D}_l . Furthermore, it is easy to see, following [12], that $\mathcal{D}_{(2)} \Psi = 0$ if and only if $|\psi\rangle_l$ is a coherent state on \mathbb{R}_θ^2 localized at $p \in \mathcal{D}_l$, with definite chirality associated with $\mathcal{D}_l \cong \mathbb{R}^2$. This can be seen by introducing the shifted creation and annihilation operators for \mathbb{R}_θ^2

$$(X_{(l)}^6 - p_6) + iX_{(l)}^7 = a^\dagger, \quad (X_{(l)}^6 - p_6) - iX_{(l)}^7 = a, \quad [a, a^\dagger] = 2\theta_{67} =: \theta, \quad (55)$$

which satisfy

$$(X_{(l)}^6 - p_6)^2 + (X_{(l)}^7)^2 = \frac{1}{2}(a^\dagger a + a a^\dagger) = \theta \left(\hat{n} + \frac{1}{2} \right). \quad (56)$$

Here, $\hat{n} = a^\dagger a$ is the number operator. We also introduce a fermionic oscillator representation for the Gamma matrices Δ_a :

$$2\alpha = \Delta_6 - i\Delta_7, \quad 2\alpha^\dagger = \Delta_6 + i\Delta_7, \quad \{\alpha, \alpha^\dagger\} = 1. \quad (57)$$

Hence, the chirality operator on \mathbb{R}_θ^2 is given by

$$\chi \equiv \chi_{67} = i\Delta_6 \Delta_7 = -2 \left(\alpha^\dagger \alpha - \frac{1}{2} \right), \quad (58)$$

acting on the spin- $\frac{1}{2}$ irreducible representation. Moreover, it is straightforward to show that

$$\Sigma_{67} = \frac{i}{4} [\Delta_6, \Delta_7] = \frac{1}{2} [\alpha, \alpha^\dagger] = \frac{1}{2} \chi. \quad (59)$$

With these tools, we can write

$$\not{D}_{(2)}\Psi = \alpha\Psi a^\dagger + \alpha^\dagger\Psi a \quad (60)$$

$$\not{D}_{(2)}^2\Psi = \Psi\theta\left(\hat{n} + \frac{1}{2}\right) - \Sigma\Psi\Theta_{(2)} = \theta\left(\Psi\hat{n} + \frac{1}{2}(1+\chi)\Psi\right). \quad (61)$$

From either equation it follows that $\not{D}_{(2)}\Psi = 0$ if and only if $\Psi\hat{n} = 0 = (1+\chi)\Psi$, which means that $\psi_{(l)}$ is a coherent state on \mathbb{R}_θ^2 localized at $p \in \mathbb{R}^2$, with definite helicity $\chi = -1$. Putting these results together, it follows that \not{D} has zero modes linking the $D0$ brane with \mathbb{R}_θ^2

$$\Psi_{p,s_l}^0 = |p, s_l\rangle_u \langle p, \downarrow|_l. \quad (62)$$

It is remarkable that this is optimally localized at $p \in \mathbb{R}^2$. However, there are two degenerate states with both chiralities $s_l = \pm 1$, corresponding to a vanishing index [19]. If p is located at some finite distance from \mathbb{R}_θ^2 , then these states are massive.

Now we switch on a Higgs field realized by the non-commutative $X_{(u)}^a$ given by the background (29). We denote the basis of $\mathcal{H}_{(u)}$ with $|\pm\rangle_u$, so that

$$X^6|\pm\rangle_u = p_\pm^6|\pm\rangle_u \quad (63)$$

with

$$p_\pm^6 = \pm \frac{r}{2}. \quad (64)$$

We claim that now \not{D}_{int} has precisely one chiral zero mode located at each p_\pm . To see this, we write down again the Dirac operator for off-diagonal fermions acting on the above states as

$$\not{D}_{\text{int}}\Psi = \not{D}_{(1)}\Psi - \not{D}_{(2)}\Psi, \quad (65)$$

where now

$$\not{D}_{(2)}\Psi = \Delta_6\tilde{X}^6\Psi + \Delta_7\Psi X_{(l)}^7, \quad \tilde{X}^6\Psi = \Psi X_{(l)}^6 - \frac{r}{2}\sigma_3\Psi, \quad (66)$$

and

$$\not{D}_{(1)}\Psi = \frac{\phi}{2}(\sigma_+\Delta_- + \sigma_-\Delta_+)\Psi, \quad \Delta_\pm = \frac{1}{2}(\Delta_4 \pm i\Delta_5). \quad (67)$$

Noting that $\{\tilde{X}^6, X_{(l)}^7\}$ satisfy the same algebra \mathbb{R}_θ^2 as $\{X_{(l)}^6, X_{(l)}^7\}$, we can introduce modified ladder operators

$$\begin{aligned} \tilde{a}\Psi &= \tilde{X}^6\Psi - i\Psi X_{(l)}^7, & \tilde{a}^\dagger\Psi &= \tilde{X}^6\Psi + i\Psi X_{(l)}^7, \\ [\tilde{a}, \tilde{a}^\dagger] &= \theta, \end{aligned} \quad (68)$$

such that

$$\not{D}_{(2)}\Psi = \alpha\tilde{a}^\dagger\Psi + \alpha^\dagger\tilde{a}\Psi \quad (69)$$

and therefore

$$\not{D}_{(2)}^2\Psi = \theta_a\left(\tilde{a}\tilde{a}^\dagger + \frac{1}{2}\right)\Psi - \Sigma\Psi\Theta_{(l)} = \theta\left(\tilde{a}\tilde{a}^\dagger\Psi + \frac{1}{2}(1+\chi)\Psi\right). \quad (70)$$

Therefore $\not{D}_{(2)}\Psi = 0$ if and only if $\tilde{a}\tilde{a}^\dagger\Psi = 0 = (1+\chi)\Psi$. This is equivalent to

$$\tilde{a}^\dagger\Psi = -\frac{r}{2}\sigma_3\Psi + \Psi(X_{(l)}^6 + iX_{(l)}^7) = 0 = \tilde{a}^\dagger\Psi, \quad (71)$$

which means that Ψ consists of coherent states localized at $X^6 = \frac{r}{2}\sigma_3$ and $X^7 = 0$. More explicitly, expanding Ψ in the appropriate basis $\{|+\rangle_u\langle n_+|_l, |-\rangle_u\langle n_-|_l\}$ (dropping helicities), where $\langle n_\pm|_l$

denotes an oscillator basis with origin p_{\pm} , it follows that the zero modes of $\mathcal{D}_{(2)}$ have the form

$$\begin{aligned}\Psi_{+(2)} &= |+, s_+\rangle_u \langle p_+, \downarrow |_l, \\ \Psi_{-(2)} &= |-, s_-\rangle_u \langle p_-, \downarrow |_l.\end{aligned}\quad (72)$$

Imposing in addition $\mathcal{D}_{(1)}\Psi = 0$ using (66) and noting that $\Delta_+|+, \uparrow\rangle_u = 0$ and $\Delta_-|-, \downarrow\rangle_u = 0$ leaves the following two zero modes of \mathcal{D}_{int} :

$$\begin{aligned}\Psi_{+L} &= |+, \uparrow\rangle_u \langle p_+, \downarrow |_l, \\ \Psi_{-R} &= |-, \downarrow\rangle_u \langle p_-, \downarrow |_l,\end{aligned}\quad (73)$$

with definite chirality L, R in \mathbb{R}^4 . These are the two chiral zero modes located at p_{\pm} expected from the picture of intersecting branes. As a check, it is straightforward to verify using (69) that the states (73) are indeed zero modes of \mathcal{D}_{int} .

To summarize, we have found that switching on ϕ , i.e., fusing the two points to a minimal fuzzy ellipsoid, lifts the degeneracy of the two polarization states at a single point. One is then left with two zero modes of opposite chirality, located at the opposite poles of the fuzzy ellipsoid.

If \mathcal{D}_l is described by some curved brane, then these would-be zero modes acquire some small mass. The associated Yukawa coupling will be proportional to the Higgs ϕ , as discussed in the next section.

4.2. $\mathbb{R}^2 \times S^2$ intersecting \mathbb{R}^2

Finally we add the missing $\mathbb{R}_{\theta(89)}^2$ to $\mathcal{D}_u = \mathbb{R}_{89}^2 \times S^2$. This is achieved simply by adding $\Delta_8 X^8 + \Delta_9 X^9$ to $\mathcal{D}_{(\text{int})}$ and $\mathcal{D}_{(1)}$, leading to an additional term

$$\beta b^\dagger \Psi + \beta^\dagger b \Psi, \quad (74)$$

where b, b^\dagger form the oscillator representation of $\mathbb{R}_{\theta(89)}^2$ and

$$2\beta = \Delta_8 - i\Delta_9, \quad 2\beta^\dagger = \Delta_8 + i\Delta_9, \quad \{\beta, \beta^\dagger\} = 1. \quad (75)$$

The additional contribution vanishes if and only if $b\Psi = 0 = \beta\Psi$, so that the above results generalize immediately. We obtain the following two zero modes of \mathcal{D}_{int} :

$$\begin{aligned}\Psi_{+L} &= | + 0, \uparrow\downarrow\rangle_u \langle p_+, \downarrow |_l, \\ \Psi_{-R} &= | - 0, \downarrow\downarrow\rangle_u \langle p_-, \downarrow |_l,\end{aligned}\quad (76)$$

with definite chirality L, R in \mathbb{R}^6 . These are the two chiral zero modes located at p_{\pm} expected from the picture of intersecting branes.

Finally, note that the Dirac equation for these fermionic would-be zero modes are not affected by $H_{(S)}^a$ due to the coherent state property $h^a X_a |0\rangle_u = 0$, except for ν_R , which could acquire a Majorana mass term via the gauge singlet $\text{tr}_N(\nu_R S)$.

4.2.1. Mirror fermions. Besides these zero modes, there are additional pairs of massive fermions (“mirror fermions”) with opposite chirality at the same intersections, coupling to the same gauge fields. The lowest ones arise from the opposite helicity of the coherent state $|+\rangle$ on the minimal S_2^2 , with eigenvalue of $\mathcal{D}_{(1)}$ of order 2ϕ . We denote those by $\tilde{\psi}$. Additional sets of ultra-massive fermions with mass of order θ arise from other helicity and oscillator states on \mathbb{R}_θ^2 . In this way, a chiral model emerges at low energies from the non-chiral underlying $\mathcal{N} = 4$ theory, with a large

hierarchy between the low-energy chiral fermions and their massive mirror partners. Such a mirror model could be phenomenologically viable provided the hierarchy is sufficiently large.

One potential problem is the fact that the tree-level mass of these lowest mirror fermions is only a factor $\sqrt{2}$ higher than the tree-level W mass, both being determined by ϕ (see Sect. 5.2). However, this is also the scale of the KK modes on the large branes, which couple to the fermions but not to the electroweak gauge fields. It then seems reasonable that quantum effects increase the mass of the mirror fermions sufficiently high above the electroweak scale.

4.3. Deformations, would-be zero modes and Yukawa couplings

4.3.1. Analytical expectation. Armed with these results for the flat case, we would like to understand the fermions arising at the intersections of the compact branes $\mathcal{D}_u \cap \mathcal{D}_l$ given by large and small fuzzy spheres (35). We expect that the qualitative features of the flat limit survive: there should be pairs of near-zero eigenmodes of \mathcal{D}_{int} , called would-be zero modes henceforth, which, due to $\mathcal{D}_{\text{int}} \Gamma^{(\text{int})} = -\Gamma^{(\text{int})} \mathcal{D}_{\text{int}}$ [cf. (A3)], decompose into states Ψ_L, Ψ_R of definite chirality, which in turn are approximately localized at the intersections p_{\pm} of the branes. However, the helicities should be determined by the local tangent planes at the intersections. We therefore expect that the following ansatz in terms of coherent states should be appropriate:

$$\begin{aligned}\Psi_{+L} &= | +0, \nearrow \searrow \rangle_u \langle p_+, \searrow |_l, \\ \Psi_{-R} &= | -0, \swarrow \swarrow \rangle_u \langle p_-, \swarrow |_l,\end{aligned}\tag{77}$$

at least if the intersection¹² is perpendicular. Here the coherent states are located at the intersection of the branes, with slightly modified helicity orientation reflecting the local geometry. The incompatible spin orientations of the pair of would-be zero modes leads to non-vanishing Yukawa couplings and eigenvalues. To gain some analytic insights, let us compute these Yukawa couplings explicitly for the above ansatz (77). Consider first

$$\begin{aligned}\text{tr}_N \Psi_{-R}^* \gamma_0 \gamma_5 \mathcal{D}_{\text{int}} \Psi_{+L} &= \text{tr}_N \left(|p_-, \swarrow \rangle_l \langle -0, \swarrow \swarrow |_u \gamma_0 \gamma_5 \mathcal{D}_{\text{int}} | +0, \nearrow \searrow \rangle_u \langle p_+, \searrow |_l \right) \\ &= \frac{1}{2} \langle p_+, \searrow |_l \langle p_-, \swarrow \rangle_l \langle -0, \swarrow \swarrow |_u | \phi(\sigma_+ \Delta_- + \sigma_- \Delta_+) | +0, \nearrow \searrow \rangle_u \\ &\approx \phi \langle p_+, \searrow |_l \langle p_-, \swarrow \rangle_l \langle -0, \swarrow \swarrow |_u | \Delta_+ | -0, \nearrow \searrow \rangle_u \\ &= \phi f_{RL}.\end{aligned}\tag{78}$$

In the last step, we observed that only the second term in $(\sigma_+ \Delta_- + \sigma_- \Delta_+)$ can give non-vanishing matrix elements between $\langle -, \swarrow |_u$ and $| +, \nearrow \searrow \rangle_u$, and evaluated the action of σ_- . The contribution from the coherent states on \mathcal{D}_l can be approximated by

$$\langle p_+, \searrow |_l \langle p_-, \swarrow \rangle_l \approx \langle p_+ | p_- \rangle_l,\tag{79}$$

since the two spin directions should be appropriately aligned, as long as \mathcal{D}_l is much larger than S_2^2 . In the flat limit this inner product would be exponentially suppressed with the distance of the two

¹² Here the classical orbit of the loci of the coherent states on the fuzzy branes is relevant. For the fuzzy sphere, this has radius $R \frac{N-1}{2}$ instead of $R \frac{\sqrt{N^2-1}}{2}$.

coherent states on \mathcal{D}_l ,

$$\langle p_+ | p_- \rangle_l \approx e^{-\frac{(p_- - p_+)^2}{R^2 N}}, \quad (80)$$

although this factor is typically very close to 1 for the compact branes under consideration. On the other hand, the contribution from the coherent states on \mathcal{D}_u can be approximated by the spin contribution only,

$$\langle -0, \nearrow \nearrow | \Delta_+ | -0, \searrow \searrow \rangle_u = \langle \nearrow \nearrow | \Delta_+ | \searrow \searrow \rangle_u. \quad (81)$$

This is non-vanishing only due to the non-alignment of the two helicities at the two intersections. Assuming that the spinor wavefunctions factorize (as in the flat case), the spinor associated with S_2^2 is given by

$$| \searrow \rangle_u = \begin{pmatrix} 1 - \frac{1}{2}\epsilon^2 \\ \epsilon \end{pmatrix}, \quad | \nearrow \rangle_u = \begin{pmatrix} \epsilon \\ 1 - \frac{1}{2}\epsilon^2 \end{pmatrix}, \quad (82)$$

where $\epsilon \sim \sin(\frac{1}{2}\vartheta)$ and ϑ measures the angle of the two distinct tangent planes $T_{p\pm}\mathcal{M}$ relative to the flat limit. This gives

$$\langle \nearrow | \Delta_+ | \searrow \rangle_u \approx \epsilon^2 \approx \frac{r^2}{4N_l^2 R_l^2}, \quad (83)$$

using (43) for the present geometry. Combining these results, we obtain the desired Yukawa couplings

$$f_{RL} \approx \frac{r^2}{4N_l^2 R_l^2} e^{-\frac{r^2}{NR^2}}. \quad (84)$$

This is clearly small for the would-be zero modes under consideration, while for the mirror fermions $\tilde{\psi}$ with reversed spin associated with the S_2^2 we would get

$$\tilde{f}_{LR} \approx e^{-\frac{r^2}{NR^2}} \quad (85)$$

due to $\langle \nearrow | \Delta_+ | \searrow \rangle_u \approx 1$. In particular, there is naturally a large hierarchy $\mathcal{O}(\epsilon^2)$ between the lowest chiral sector corresponding to the standard model, and the first series of massive mirror fermions. Moreover, these quantities are accessible, both to (refined) analytical considerations and to numerical methods. Of course they will also be subject to quantum corrections, which are out of the scope of the present paper. One may hope that these quantum corrections help to increase the separation of the mirror fermions from the electroweak W, Z bosons, as discussed below.

4.3.2. Numerical results. Some aspects of the theoretical expectations derived in the previous subsection can be verified numerically. We compute the eigenvalues of the internal Dirac operator for the off-diagonal spinors connecting the two branes, and identify them with the Yukawa couplings of their chiral components. Restricting to a regime where the Planck cells of the larger fuzzy spheres are greater than r , we can neglect the Gaussian factor in (84), and obtain for the lowest eigenvalue

$$\lambda_0 \approx \frac{\phi r^2}{4N_l^2 R_l^2}, \quad (86)$$

and for the next eigenvalue, i.e., the mirror fermions,

$$\lambda_1 \approx \phi. \quad (87)$$

In Figs. 4, 5, and 6, we see that these estimates agree quite well with numerical results—the red lines show the expectations from (86) and (87). Also, the next eigenvalue λ_2 is shown. Furthermore, we see that one can produce large hierarchies for moderate N .

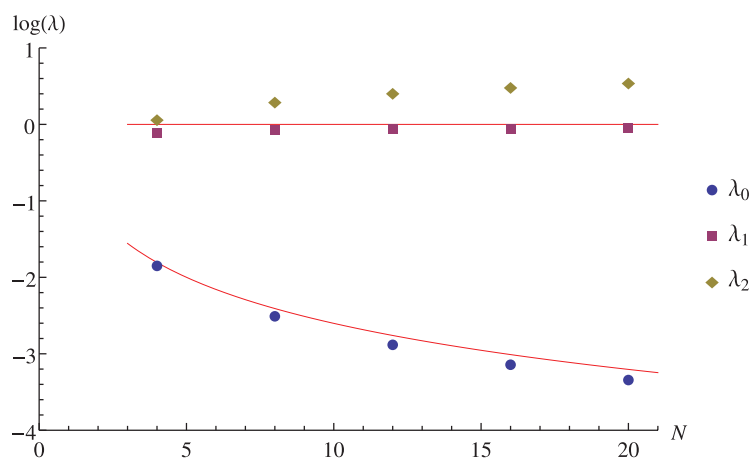


Fig. 4. Lowest eigenvalues as a function of N , for $N_{l,u} = N$, $R_{l,u} = R'_{l,u} = 1$, and $r = \phi = 1$, with the theoretical expectations (86) and (87).

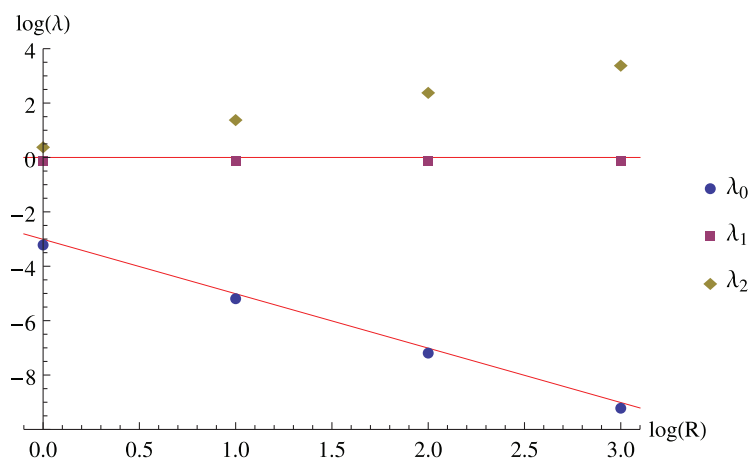


Fig. 5. Lowest eigenvalues as a function of R , for $N_{l,u} = 16$, $R_{l,u} = R'_{l,u} = R$, and $r = \phi = 1$, with the theoretical expectations (86) and (87).

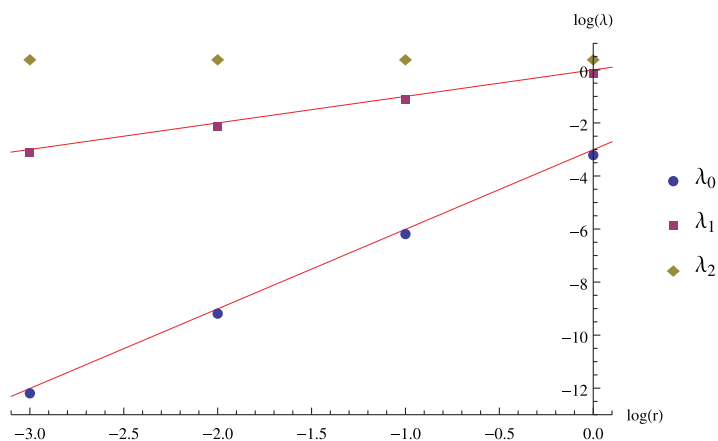


Fig. 6. Lowest eigenvalues as a function of r , for $N_{l,u} = 16$, $R_{l,u} = R'_{l,u} = 1$, and $r = \phi$, with the theoretical expectations (86) and (87).

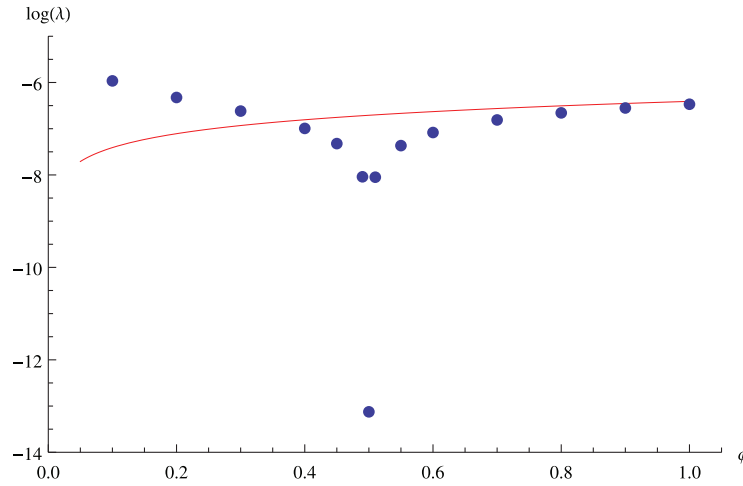


Fig. 7. Lowest eigenvalue as a function of ϕ , for $N_{l,u} = 8$, $R_{l,u} = R'_{l,u} = 1$, and $r = 1$, with the theoretical expectation (86).

However, varying ϕ while keeping r fixed leads to a dramatic deviation from the expectation (86), as shown in Figure 7 (note that the scale is logarithmic): There is a very pronounced minimum of λ_0 at $\phi = r/2$ (which is excluded by the equation of motion in Sect. 3.4). This is also seen for other choices of N , R , so it seems to be a universal behavior. From geometrical considerations, one would rather expect $\phi \approx \frac{1}{\sqrt{2}}r$ to be special, as then the branes intersect orthogonally. Hence, a complete understanding of the Yukawa couplings is missing, but the generation of a large gap between the lowest and the next eigenvalue of \mathcal{D}_{int} is certainly possible.

For our arguments, it is crucial that the lowest eigenstates, when projected to a definite six-dimensional chirality, are very well localized at $\pm(0, 0, 1)$ on S^2_2 , and are essentially eigenvectors of $\chi_{45} = 2\Sigma_{45}$, the chirality operator corresponding to the 4–5 plane.¹³ From (82), we expect the expectation value of $1 - \Sigma_{45}$ in the lowest eigenstate ψ_0 to be roughly

$$\langle \psi_0 | (1 - \Sigma_{45}) | \psi_0 \rangle \approx 2\epsilon^2 \approx \frac{r^2}{2N_l^2 R_l^2}. \quad (88)$$

Figure 8 shows the expectation value of $1 - \sigma_3$ and $1 - \Sigma_{45}$ in the lowest eigenstate as a function of $R = R_{l,u} = R'_{l,u}$. Also, the expectation (88) is plotted.¹⁴ We see that the deviation from being an eigenstate indeed decreases for increasing R .

4.4. Gauginos

Besides the fermions arising at the interactions of the various branes, fermions also arise in the diagonal blocks, as functions on the corresponding branes. They can be viewed as (generalized) gauginos, i.e. superpartners of the gauge bosons or scalar fields. All these fermions are non-chiral, i.e. both chiral sectors couple identically to the gauge and scalar fields. This includes the gluinos, binos, winos, Higgsinos, etc. Note that the gauginos corresponding to $U(1)_B$ and $U(1)_l$ are neutral under the full standard model gauge group, and therefore decouple at low energies. As usual, they

¹³ Note that they are then also essentially eigenvectors of $\chi_{67}\chi_{89}$. Projecting on the eigenspaces of, say, χ_{89} , one obtains a vector which is essentially an eigenvector of Σ_{45} , Σ_{67} , and Σ_{89} .

¹⁴ Note that our ansatz was that ψ_0 is an eigenstate of σ_3 corresponding to S^2_2 , so the above discussion does not give a prediction for the expectation value of $1 - \sigma_3$.

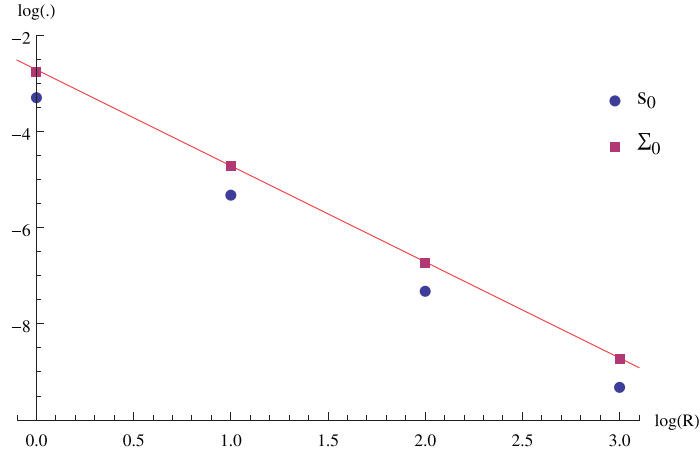


Fig. 8. Expectation values of $s = 1 - \sigma_3$ and $\Sigma = 1 - \Sigma_{45}$ in the lowest eigenvalue as a function of R , for $N_{l,u} = 16$, $R_{l,u} = R'_{l,u} = R$, and $r = \phi = 1$, with the theoretical expectation (88).

may be considered as dark matter candidates. The gluinos and other gauginos are expected to get radiative mass. Furthermore, there are towers of higher KK modes for all these fermions. However, note that the present backgrounds are far from supersymmetric, since, e.g., the scalar superpartners of the standard model fermions have tree-level mass of order $\theta \sim R^2 N$.

4.5. Fermion masses and Yukawas

In this section, we show that the four-dimensional masses of the would-be zero modes are given by the corresponding Yukawa couplings. As recalled in Appendix A, see also (9), the Dirac operator can be written as

$$\not{D}\Psi = \left(\not{D}_4 + \rho^{\frac{1}{2}} \gamma_5 \not{D}_{\text{int}} \right) \Psi. \quad (89)$$

Here,

$$\not{D}_4 = \tilde{\gamma}^\mu (i\partial_\mu + [\mathcal{A}_\mu, \cdot]) \quad (90)$$

is the massless four-dimensional Dirac operator on \mathbb{R}_θ^4 with $\tilde{\gamma}^\mu = \rho^{\frac{1}{2}} \theta^{\nu\mu} \gamma_\nu$, and \not{D}_{int} is defined in terms of the X^a . Now consider a pair of eigenspinors ψ_\pm of the internal Dirac operator

$$\rho^{\frac{1}{2}} \not{D}_{\text{int}} \psi_\pm = \pm m \psi_\pm, \quad \psi_\pm = \psi_\uparrow \pm \psi_\downarrow \quad (91)$$

in terms of two chirality eigenstates (such as our would-be zero modes), with $\Gamma^{(\text{int})} \psi_\uparrow = \pm \psi_\uparrow$ to be specific. Then $\rho^{\frac{1}{2}} \not{D}_{\text{int}} \psi_\uparrow = m \psi_\downarrow$, corresponding to a Yukawa coupling m . Now consider a 32-component spinor whose internal components consist of the above two internal helicity states,

$$\Psi = \chi_1 \otimes \psi_\uparrow + \chi_2 \otimes \psi_\downarrow, \quad (92)$$

where $\chi_{1,2}$ are Dirac spinors of $SO(3, 1)$. This can be represented as an auxiliary eight-component spinor consisting of the two Dirac spinors of $SO(3, 1)$ only,

$$\Psi \cong \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad (93)$$

and the ten-dimensional Dirac equation can be written as

$$0 = \not{D}\Psi = \begin{pmatrix} \not{D}_4 & \gamma_5 m \\ \gamma_5 m & \not{D}_4 \end{pmatrix} \Psi. \quad (94)$$

This has solutions with four-dimensional mass m , since

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_4^2 + m^2 & 0 \\ 0 & \mathcal{D}_4^2 + m^2 \end{pmatrix}, \quad (95)$$

noting that $\{\gamma^5, \tilde{\gamma}^\mu\} = 0$.

Finally, we show that the scalar Higgs S (45) does not affect the fermionic would-be zero modes. It contributes to the internal Dirac operator as follows:

$$\mathcal{D}_{\text{int}}^{(S)}\psi = \Delta_a[X_{(S)}^a, \psi] = 2h\beta^+[S, \psi] + 2h\beta[S^\dagger, \psi], \quad (96)$$

recalling the definition of β in Sect. 4.2. This vanishes due to $\beta\psi = 0$ for the would-be zero modes, using the explicit form of S .

5. Symmetry breaking and four-dimensional fields

Spelling out all the branes, the background (35) is given by

$$\begin{aligned} X^4 + iX^5 &= \begin{pmatrix} R'_u L_3 & 0 & 0 & \phi \mathbb{1} \\ 0 & R'_d L_3 & \phi \mathbb{1} & 0 \\ 0 & 0 & R'_d L_3 & 0 \\ 0 & 0 & 0 & R'_u L_3 \end{pmatrix}, \\ X^6 + iX^7 &= \begin{pmatrix} -\frac{r}{2} \mathbb{1} & & & \\ & -\frac{r}{2} \mathbb{1} & & \\ & & \frac{r}{2} \mathbb{1} & \\ & & & \frac{r}{2} \mathbb{1} \\ & & & & RK_+ \\ & & & & & RK_+ \mathbb{1}_3 \end{pmatrix}, \\ X^8 + iX^9 &= \begin{pmatrix} RL_+ & & & & \\ & RL_+ & & & \\ & & RL_+ & & \\ & & & RL_+ & 2hS \\ & & & & 0 \\ & & & & & 0_3 \end{pmatrix}. \end{aligned} \quad (97)$$

We also included the Higgs singlet S , which connects the branes \mathcal{D}_u and \mathcal{D}_l .

We want to understand the bosonic modes which arise as fluctuations $X^a \rightarrow X^a + \mathcal{A}^a$ about the above background. In general, such fluctuations on a fuzzy brane \mathcal{K}_N can be written as a finite sum

$$\mathcal{A}(x, y) = \sum_{lm} \mathcal{A}_{lm}(x) Y^{lm}, \quad (98)$$

where $Y^{lm} \in \text{Mat}(N, \mathbb{C})$ stands symbolically for the harmonics of $\square = [X^a, [X^a, \cdot]]$ on \mathcal{K}_N . This applies equally to scalar fields, gauge fields, and the gauginos. It provides a geometric interpretation of the matrix-valued fields on \mathbb{R}^4 in terms of towers of massive Kaluza–Klein modes on \mathcal{K}_N . On the fuzzy sphere S_N^2 , this KK tower arises at roughly equidistant masses determined by the eigenvalues of \square , with lowest non-trivial eigenvalue $\sim R^2$. Therefore, at low energies, it suffices to keep only the

massless modes $\sim \mathbb{1}$ on the S_N^2 . Then the $\mathcal{A}(x, y)$ can be viewed as functions on \mathbb{R}^4 taking values in the above space of 8×8 matrices. In particular, the stack of three coincident \mathcal{D}_B branes gives rise to the massless gluons with unbroken $U(3) = SU(3)_c \times U(1)_B$ gauge symmetry, as well as an associated finite tower of massive KK modes. On the other hand, the KK tower on the minimal branes is very short, and contains in particular the electroweak Higgs, the Z boson, and the B_5 and C_μ bosons, as discussed below.

5.1. Hierarchical symmetry breaking

To determine the masses of the low-energy gauge bosons explicitly, it is useful to first replace the two minimal fuzzy spheres $S_{N=2}^2$ by a stack of four coincident D0 branes $X_{(1)}^a = p^a \sum_{i=1}^4 |i\rangle\langle i|$ located at some point p^a on the coincident $\mathcal{D}_l \cong \mathcal{D}_B$ branes described by $\mathbb{1}_4 \otimes X_{(2)}^a$. This background admits a $U(4) \times U(4)$ symmetry. Now we switch on a non-vanishing singlet Higgs $S \sim |i\rangle\langle p|_l$, where $|p\rangle_l$ is a coherent state on $\mathcal{D}_{(l)}$ located at the D0 branes p^a . Since S has rank one, it breaks the symmetry to the commutant $U(3)_B \times U(3) \times U(1)_S$, where $U(1)_S$ acts diagonally on $|i\rangle \oplus |p\rangle_l$. This $U(1)_S$ can be traded for $U(1)_{\text{tr}}$, which has a clear geometric interpretation. We assume here that this breaking happens at a high scale, and restrict ourselves to the commutant of S from now on. The fermionic modes on such a background are still non-chiral.

Next, we introduce the long axis along X^6 of the electroweak ellipsoids by turning on $r > 0$. This breaks the above symmetry further to the commutant of X^6 (in $U(3)_B \times U(3) \times U(1)_{\text{tr}}$). The bosons C_μ associated with this breaking will be discussed below. Using the explicit form (97), this commutant is given by $U(3)_B \times SU(2)_L \times U(1)_Y \times U(1)_5 \times U(1)_{\text{tr}}$, with generators¹⁵

$$t_{\pm,3} = \frac{1}{2} \begin{pmatrix} \sigma_{\pm,3} I_{N_1} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \\ & & & & 0 \end{pmatrix}, \quad t_Y = \begin{pmatrix} 0_2 & & & & \\ & -\mathbb{1}_{N_1} & & & \\ & & \mathbb{1}_{N_1} & & \\ & & & \mathbb{1}_{N_2} & \\ & & & & -\frac{1}{3}\mathbb{1}_{N_2} \end{pmatrix},$$

$$t_5 = \begin{pmatrix} \mathbb{1}_2 \otimes \mathbb{1}_{N_1} & & & & \\ & -\mathbb{1}_{N_1} & & & \\ & & -\mathbb{1}_{N_1} & & \\ & & & -\mathbb{1}_{N_2} & \\ & & & & \frac{1}{3}\mathbb{1}_{N_2} \end{pmatrix}, \quad t_{(s)\alpha} = \begin{pmatrix} 0_2 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \lambda_\alpha \mathbb{1}_{N_2} \end{pmatrix}. \quad (99)$$

Here we assume that $\mathcal{D}_{l,B}$ is represented on \mathbb{C}^{N_2} , and $\lambda_\alpha \in \mathfrak{u}(3) = \mathfrak{su}(3) \oplus \mathfrak{u}(1)_B$. Note that t_5 acts as

$$[t_5, \cdot] \cong B - l + \gamma_5 \quad (100)$$

on the fermionic zero modes. It is therefore anomalous and expected to disappear from the low-energy spectrum, along with $U(1)_B$. This leaves exactly the gauge group of the standard model $SU(3)_c \times SU(2)_L \times U(1)_Y$, extended by the anomalous $U(1)_B$, $U(1)_5$, and the geometric $U(1)_{\text{tr}}$. The $U(1)_5$ is also broken by the electroweak Higgs, as elaborated below.

¹⁵ Here, I_{N_1} indicates the identification $\mathbb{C}^{N_u} \cong \mathbb{C}^{N_d}$ of the Hilbert spaces of $S_{N_u}^2$ and $S_{N_d}^2$, assuming $N_u = N_d = N_1$.

Finally, we switch on the electroweak Higgs $\phi_{u,d}$, so that the four $D0$ branes expand to form two fuzzy ellipsoids S_2^2 . Then the symmetry breaks down as desired to $SU(3)_c \times U(1)_Q \times U(1)_B \times U(1)_{\text{tr}}$, with charge generator

$$t_Q = t_3 + \frac{1}{2}t_Y = \frac{1}{2} \left(\mathbb{1}_u - \mathbb{1}_d + \mathbb{1}_l - \frac{1}{3}\mathbb{1}_B \right). \quad (101)$$

Here, $U(1)_B$ is anomalous, and $U(1)_{\text{tr}}$ is a geometric mode associated with gravity.

As a check, the unbroken gauge group of the above brane configuration can alternatively be obtained as follows. Consider first the background without S , given by two coincident branes $\mathcal{D}_u \simeq \mathcal{D}_d$ and four coincident branes $\mathcal{D}_l \simeq \mathcal{D}_B$. This background has an unbroken $U(2) \times U(4)$ gauge symmetry. Now we switch on the scalar Higgs S . This breaks the symmetry to its commutant $SU(3)_c \times U(1)_Q \times U(1)_B \times U(1)_{\text{tr}}$.

5.2. Four-dimensional gauge bosons and masses

We recall the four-dimensional form of the effective action (9) of the matrix model in our brane background. To obtain the proper coupling constants for the corresponding gauge fields, we introduce canonically normalized generators with $\text{tr}_N(\tilde{t}_i \tilde{t}_j) = \frac{1}{2} \delta_{ij}$, via

$$\begin{aligned} t_{\pm 3} &= c_L \tilde{t}_{\pm,3}, & c_L^2 &= N_1 \\ t_Y &= c_Y \tilde{t}_Y, & c_Y^2 &= 2 \left(2N_1 + \frac{4}{3}N_2 \right) \\ t_5 &= c_5 \tilde{t}_5, & c_5^2 &= 2 \left(4N_1 + \frac{4}{3}N_2 \right) \\ t_\alpha &= c_S \tilde{t}_\alpha, & c_S^2 &= N_2. \end{aligned} \quad (102)$$

The point is that the \tilde{t}_α act on the full Hilbert space of the matrix model and satisfy rescaled commutation relations, while the t_α act on the reduced Hilbert space of physical fermionic states as in the standard model. To identify the low-energy gauge couplings, we write the gauge fields in two ways using (99):

$$\begin{aligned} \mathcal{A} &= g_{\text{YM}} \left(W_- \tilde{t}_+ + W_+ \tilde{t}_- + W_3 \tilde{t}_3 + B \tilde{t}_Y + B_5 \tilde{t}_5 + A_\alpha \tilde{t}_\alpha \right) \\ &= g(W_- t_+ + W_+ t_- + W_3 t_3) + \frac{1}{2} g' B t_Y + g_5 B_5 t_5 + g_S A_\alpha t_\alpha, \end{aligned} \quad (103)$$

where

$$g = \frac{g_{\text{YM}}}{c_L}, \quad g_S = \frac{g_{\text{YM}}}{c_S}, \quad \frac{1}{2} g' = \frac{g_{\text{YM}}}{c_Y}, \quad g_5 = \frac{g_{\text{YM}}}{c_5}. \quad (104)$$

The effective standard model coupling constants in the second line are identified from the covariant derivative on the fermions

$$\begin{aligned} iD_\mu \psi &= \theta_{\mu\nu}^{-1} [X^\nu, \psi] = (i\partial_\mu + [\mathcal{A}_\mu, \cdot]) \psi \\ &\stackrel{!}{=} \left(i\partial_\mu + g W_i t_i + \frac{g'}{2} B t_Y + g_S A_\alpha t_\alpha + g_5 B_5 t_5 \right) \psi \end{aligned} \quad (105)$$

on the fermionic would-be zero modes. Since the relevant fermionic (would-be) zero modes are made of one-dimensional (coherent) states in the internal Hilbert space, the term $\text{tr}_N \bar{\psi} \tilde{\gamma}^\mu iD_\mu \psi$ in (9) reduces to the appropriate Lagrangian for the four-dimensional fermions in the standard model, without any extra factors coming from tr_N . We can therefore identify the gauge fields W_i , B , etc.

with those of the standard model, where g is the $SU(2)_L$ coupling constant, g' is the $U(1)_Y$ coupling constant, g_S is the strong coupling constant, and g_5 the one associated with $U(1)_5$. These tree-level couplings apply at very high energies. The kinetic terms of these gauge fields have the standard normalization, and by gauge invariance their full action must be

$$S_{\text{YM}} = - \int d^4x \frac{1}{4g_{\text{YM}}^2} \text{tr}_N(\mathcal{F}\mathcal{F})_G = - \int d^4x \frac{1}{4} \text{tr}_{\text{red}}(FF)_G + \dots \quad (106)$$

Here, tr_{red} is the trace in the adjoint representation of the reduced low-energy gauge group $\oplus_i \mathfrak{g}_i$ generated by the t_i , with gauge fields A , W , etc. corresponding to the (extended) standard model; for example, the contributions of the gluons is $F_S = \partial_\mu A_\nu - \partial_\nu A_\mu + g_S[A_\mu, A_\nu]$. The correct normalization for the interacting terms follows from gauge invariance, and can be verified directly, e.g. for the gluons, where $\frac{1}{4g_{\text{YM}}^2} \text{tr}_N(\mathcal{F}\mathcal{F})_G$ gives rise to

$$\frac{1}{4g_{\text{YM}}^2} \text{tr}_N(g_{\text{YM}}^4 [A\tilde{t}, A\tilde{t}][A\tilde{t}, A\tilde{t}])_G = \frac{1}{4} \text{tr}_{\text{red}} \left(g_S^2 [At, At][At, At] \right)_G. \quad (107)$$

Finally, consider the action for the scalar fields Φ^a , which describe the internal branes and contain in particular the Higgs. Their kinetic term

$$- \int d^4x \frac{1}{2} \text{tr}_N (D_\mu \Phi^a D^\mu \Phi^a) \quad (108)$$

leads as usual to SSB of the gauge fields. The most interesting part is the electroweak symmetry breaking, induced by the minimal fuzzy ellipsoids S_N^2 . Let us elaborate their effect on the low-energy fields. These terms arise from

$$\begin{aligned} S_\phi[A] &= -\frac{1}{2} \int d^4x G^{\mu\nu} \text{tr}_N (D_\mu \Phi^4 D_\nu \Phi^4 + D_\mu \Phi^5 D_\nu \Phi^5) \\ &= -\frac{1}{2} \int d^4x G^{\mu\nu} \text{tr}_N ((D_\mu \Phi^+)^{\dagger} D_\nu \Phi^+). \end{aligned} \quad (109)$$

In view of (97), it is natural to organize the non-vanishing entries of $X^+ = X^4 + iX^5$ in terms of “effective” Higgs doublets

$$H_d = \begin{pmatrix} 0 \\ \phi_d \end{pmatrix}, \quad H_u = \begin{pmatrix} \phi_u \\ 0 \end{pmatrix}, \quad (110)$$

with $Y(H_d) = 1$ as in the standard model, and $Y(H_u) = -1$ as in the MSSM. Their eigenvalues under $[t_5, \cdot]$ are $+2$. The scalar fields ϕ have dimensions L^{-1} in this section, absorbing the scale factor $\frac{\Lambda_0^2}{\pi}$ (10) in their definition. Moreover, we will set $\phi_u = \phi_d$ for the VEVs due to the relation (37), which implies

$$\tan \beta = \frac{\phi_u}{\phi_d} = 1. \quad (111)$$

Note that this tree-level relation holds at very high energies, before integrating out any of the $\mathcal{N} = 4$ fields. Then $S_\phi[A]$ takes the standard form of a mass term arising from the covariant derivative of a two-component Higgs H_d in the standard model, supplemented by the contribution from a second

two-component Higgs H_u ,

$$S_\phi[A] = -\frac{1}{2} \int d^4x G^{\mu\nu} \text{tr}_N \left((D_\mu H_d)^\dagger D_\nu H_d + (D_\mu H_u)^\dagger D_\nu H_u \right). \quad (112)$$

Here,

$$\begin{aligned} DH_d &= [\mathcal{A}, H_d] = g W_a t_a H_d + \frac{1}{2} (g' B + 2g_5 B_5) H_d \\ &= \frac{\phi}{2} \begin{pmatrix} g(W_1 + iW_2) \\ -gW_3 + g'B + 2g_5 B_5 \end{pmatrix} = \frac{\phi}{2} \begin{pmatrix} g(W_1 + iW_2) \\ -g_Z Z + 2g_5 B_5 \end{pmatrix}, \\ DH_u &= g W_a t_a H_d + \frac{1}{2} (-g' B + 2g_5 B_5) H_u \\ &= \frac{\phi}{2} \begin{pmatrix} g(W_1 - iW_2) \\ gW_3 - g'B + 2g_5 B_5 \end{pmatrix} = \frac{\phi}{2} \begin{pmatrix} g(W_1 - iW_2) \\ g_Z Z + 2g_5 B_5 \end{pmatrix}. \end{aligned} \quad (113)$$

The Z boson is identified as the combination of W_3 and B , which acquires a mass

$$g_Z Z = g W_3 - g' B. \quad (114)$$

On the other hand, the last form of (101) guarantees that t_Q does not couple to the Higgs. The masses are obtained from¹⁶

$$S_\phi[A] = - \int d^4x \text{tr}_N \left(\frac{\phi^2}{4} g^2 (W_1^2 + W_2^2) + \frac{\phi^2}{4} (g^2 + g'^2) Z^2 + \phi^2 g_5^2 B_5^2 \right). \quad (115)$$

We can then read off the W and Z bosons' masses in the high-energy regime, taking into account a factor N_1 from tr_N :

$$\begin{aligned} m_W^2 &= \frac{1}{2} N_1 g^2 \phi^2, \quad m_Z^2 = \frac{1}{2} N_1 (g^2 + g'^2) \phi^2 \\ m_5^2 &= 2 N_1 g_5^2 \phi^2. \end{aligned} \quad (116)$$

The $U(1)_5$ is anomalous at low energies, hence it is expected to disappear from the low-energy spectrum by some Stückelberg-type mechanism; cf. [35–37]. The photon and the Z -boson are then identified as usual:

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} \sin \theta_W & \cos \theta_W \\ \cos \theta_W & -\sin \theta_W \end{pmatrix} \begin{pmatrix} W_3 \\ B \end{pmatrix} = \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g' & g \\ g & -g' \end{pmatrix} \begin{pmatrix} W_3 \\ B \end{pmatrix}. \quad (117)$$

This gives the Weinberg angle

$$\tan \theta_W = \frac{g'}{g} = \frac{2c_L}{c_Y} = \frac{1}{\sqrt{1 + \frac{2N_2}{3N_1}}} \quad (118)$$

and

$$\sin^2 \theta_W = \frac{1}{1 + \frac{g'^2}{g^2}} = \frac{1}{2 + \frac{2N_2}{3N_1}}. \quad (119)$$

For $N_1 = N_2$ this gives $\sin^2 \theta_W = 3/8$ and $g_5 = g$, as in the $SU(5)$ GUT.

Similarly, we can compute the mass of the gauge bosons C_μ associated with the breaking of $SU(3) \rightarrow SU(2)_L \times U(1)$ due to $r > 0$. A typical generator $t_C \sim |+\rangle_d \langle -|_{u,d} \mathbb{1}_{N_1}$ links the standard model fermions to the first massive mirror fermions, such as $\tilde{d}_R \leftrightarrow d_L, u_L$ or $\tilde{e}_R \leftrightarrow \nu_L, e_L$.

¹⁶ The contraction of the vector fields with $G^{\mu\nu}$ is understood.

Since t_C relates different eigenvalues of X_6 , we have $[X_6, t_C] = \pm r t_C$. There will also be a mass contribution from the Higgs ϕ in the same way as the W bosons, leading to a mass term

$$S_{X_6}[C] = -\frac{1}{2} \int d^4x \operatorname{tr}_N \left(g^2 \left(r^2 + \frac{\phi^2}{4} \right) C^\mu C_\mu \right). \quad (120)$$

This is larger than the W_\pm mass, assuming $r > \frac{\phi}{2}$.

Next, consider the contribution to the gauge boson masses from the singlet Higgs S via $X_{(S)}^a = h(e^8 + ie^9)S + h.c.$. Recalling (45) and (46), the relevant terms in the action are

$$\begin{aligned} S_S[A] &= -\frac{1}{2} \int d^4x \operatorname{tr}_N \left(\sum_{a=8}^9 D_\mu \Phi_{(S)}^a D^\mu \Phi_{(S)}^a \right) \\ &= -2 \int d^4x h^2 \operatorname{tr}_N (D_\mu S^\dagger D^\mu S), \end{aligned} \quad (121)$$

dropping possible fluctuations of S here. This gives a mass to every gauge field coupling to S . It breaks the lepton number $U(1)_l$, and in the absence of r , ϕ it breaks the electroweak $U(4)$ to $U(3)$ [subsequently broken to $SU(2)_L$ by (120)]. In particular, S breaks $U(1)_{B-l}$, which is anomaly free and would otherwise lead to an unphysical massless gauge boson, whereas $U(1)_l$ is anomalous and expected to disappear from the low-energy spectrum anyway.

Finally, the fermion masses for the off-diagonal fermions arise from

$$\begin{aligned} \int d^4x g_{YM} \operatorname{tr}_N \bar{\psi} \Gamma^a [\Phi_a, \cdot] \psi &= 2 \int d^4x g_{YM} \operatorname{tr}_N \bar{\psi}_{12} \Gamma^a [\Phi_a, \cdot] \psi_{12} \\ &= 2 \int d^4x g_{YM} f_\psi \phi \bar{\psi}_{12} \psi_{12}, \end{aligned} \quad (122)$$

taking into account the factor 2 from (A13), which also enters the kinetic term. Here, f_ψ is the Yukawa coupling for the fermion under consideration, as discussed in Sect. 4.3. The trace tr_N gives no extra factor since the fermions are made from coherent states. Therefore, the fermion mass is given by

$$m_\psi \sim g_{YM} \phi f_\psi, \quad (123)$$

where f_ψ is the corresponding Yukawa coupling. For the first series of mirror fermions we found $\tilde{f}_\psi \approx 1$, so that their tree-level (!) mass is about $\sqrt{2}$ times the W mass. In contrast, the standard model fermions have much smaller Yukawas.

At first sight, the low scale of the mirror fermions seems very bad. However, keep in mind that we merely computed the tree-level masses here,¹⁷ valid at high energies in the $\mathcal{N} = 4$ regime. At lower energies, the Yukawa couplings will be subject to quantum corrections. For example, an effective factor $\alpha > 1$ in front of the internal Dirac operator $\alpha \mathcal{D}_{(\text{int})}$ raises the fermion masses without affecting the boson masses, thus increasing the gap between the electroweak scale and the first mirror fermions. More specifically, since the fermions are given by localized (coherent) states on the large branes S_N^2 , they couple to all the massive KK gauge and scalar fields arising on these branes. These KK modes start at a scale set by R , which is comparable to the scale of the first mirror fermions $\tilde{\psi}$ by (37). Therefore they will contribute significantly to the Yukawa couplings. In contrast, these KK modes do not contribute to the mass of the W , Z gauge bosons, because these are $\propto \mathbb{1}$ on the large branes.

¹⁷ The term V_{quant} (32) in the effective potential does not affect the mass of the fermions and gauge bosons.

This should magnify the gap between the electroweak scale and the first mirror fermions, and one may hope that the model can become phenomenologically viable in this way.

In any case, the model clearly predicts mirror fermions with opposite chirality at not very high energies. These mirror fermions interact with the standard model gauge bosons, and can decay into the standard model fermions via the heavy gauge bosons C_μ . More quantitative statements would require computing quantum effects.

5.3. Moduli and the Higgs potential

The action for the geometrical moduli ϕ, r, R_i is obtained from the modified matrix model action (32) as

$$S[r, \phi, R] = - \int d^4x (V_{\text{quant}}(r, \phi, R) + V_{\text{int}}(r, \phi, R)),$$

$$V_{\text{quant}}(r, \phi, R) = \rho f \left(\pi \rho^{\frac{1}{2}} g_{\text{YM}}^{-1} \left((R'_u)^2 + 2R_u^2 \right) 2c_{N_u} + (r_u^2 + 2\phi_u^2) N_u c_2 + (R'_l{}^2 + 2R_l^2) c_{N_l} \right),$$

$$V_{\text{int}}(r, \phi, R) = \frac{\rho^2}{4g_{\text{YM}}^2} 2 \left((2R'_u{}^2 R_u^2 + R_u^4) 2c_{N_u} + (2\phi_u^2 r_u^2 + \phi_u^4) N_u c_2 + (2R'_l{}^2 R_l^2 + R_l^4) c_{N_l} \right),$$

where

$$c_N = \text{tr}_N L_3^2 = \sum_{m=-(N-1)/2}^{(N-1)/2} m^2 = \frac{1}{12} N(N^2 - 1).$$

We have

$$V(r, \phi + \delta\phi, R) = V(r, \phi, R) + \left(2\pi \rho^{\frac{3}{2}} g_{\text{YM}}^{-1} \phi f' + \rho^2 g_{\text{YM}}^{-2} (\phi r^2 + \phi^3) \right) N_u \delta\phi$$

$$+ \frac{1}{2} \left(2\pi \rho^{\frac{3}{2}} g_{\text{YM}}^{-1} f' + 4\pi^2 \rho^2 g_{\text{YM}}^{-2} \phi^2 f'' + \rho^2 g_{\text{YM}}^{-2} (r^2 + 3\phi^2) \right) N_u \delta\phi^2 + \mathcal{O}(\delta\phi^3).$$

The coefficient of the term of order $\delta\phi$ vanishes, by the equation of motion (37). The remainder can be simplified, and comparison with the kinetic term yields the following mass squared for the fluctuations $\delta\phi$ (here we introduced physical units):

$$m^2 = 2g_{\text{YM}}^2 \phi^2 (1 + 2\pi^2 f'').$$

This is positive and somewhat larger than m_W^2 , unless f'' is too negative.

There is another interesting set of low-energy perturbations, given by Goldstone bosons of the global $SO(6)$ symmetry acting uniformly on all matrices, corresponding to local rotations of the matrix background. This affects only the trace- $U(1)$ sector of the model and leads to metric perturbations related to the effective or “emergent” gravity on \mathbb{R}_θ^4 , as elaborated in [40] for a similar type of background.

5.4. Further aspects

5.4.1. Anomalies and massive gauge fields. In the present type of background (as in analogous brane configurations in string theory [22–28]), a $U(1)$ gauge symmetry arises on each brane, some of which are anomalous at low energies. This does not signal an inconsistency, since the fundamental $U(N)$ gauge symmetry is anomaly free. Rather, it indicates that the corresponding anomalous gauge bosons acquire a mass and disappear from the low-energy physics. This topic has been discussed extensively in the literature; see, e.g., [37], or [35,36] in a closely related context, based on a type of Stückelberg mechanism with an axion.

In fact, axion-like fields appear in non-commutative gauge theory via the term $\int \eta(x) F \wedge F$, where the “axion” is realized by the geometric field $\eta(x) = Gg$ in the picture of emergent gravity [8,9,44,45]. This should be related to the Chern–Simons terms arising in the D-brane action in string theory. The precise origin of such mass terms in the present context should be clarified.

In particular, baryon number $U(1)_B$ is such an anomalous gauge symmetry. It should still provide protection from proton decay, in contrast to many grand unified models. This is important in view of the highly populated spectrum of fields at intermediate energies.

5.4.2. Generations. Additional generations can arise if the large fuzzy spheres S_N^2 in either $\mathcal{D}_{u,d}$ or $\mathcal{D}_{l,c}$ are replaced by stacks of spheres with slightly different parameters. On the other hand, we have seen that there are in fact two separated intersection regions contributing to, e.g., $\mathcal{D}_u \cap \mathcal{D}_l$. This would also manifest itself as doubling of generations, which is actually unwelcome as it would imply an even number of generations. However, one of these intersection regions could be removed in principle.

5.4.3. Right-handed neutrinos. One clear prediction of our solution is the presence of right-handed neutrinos ν_R , which acquire a Dirac mass term determined by the corresponding Yukawa coupling. In addition, it seems plausible that a Majorana mass term (31) is induced by quantum effects. This aspect should be studied in more detail. For a survey on the phenomenological aspects of right-handed neutrinos we refer to the recent review [46].

6. Discussion and conclusion

We have shown that the IKKT model can behave very similarly to the standard model at low energies, for suitable backgrounds. We provided such backgrounds consisting of branes in the internal space, which are solutions of the matrix model, assuming a suitable stabilizing term in the effective potential and a non-linear stabilization of the singlet Higgs. Our results also apply to $\mathcal{N} = 4$ $SU(N)$ SYM with sufficiently large N , challenging the standard lore that $\mathcal{N} = 4$ SYM can only be a “spherical cow” approximation to realistic gauge theories. We recover the chiral fermions of the standard model with the correct quantum numbers coupling appropriately to the electroweak gauge fields. Right-handed neutrinos arise, as well as towers of massive Kaluza–Klein modes of other fields, ultimately completing the full $\mathcal{N} = 4$ spectrum at very high energies. Our results are supported by numerical computations of the spectrum of low eigenvalues of the internal Dirac operator \mathcal{D}_{int} , verifying also the chirality and localization properties of the corresponding fermionic modes.

One clear prediction is the existence of mirror fermions at intermediate energies, which can decay to standard model fermions via massive gauge bosons. The mass of the lowest mirror fermions is rather low at tree level (about $\sqrt{2}$ times the W mass, which obviously would not be realistic); however, it seems likely that quantum effects raise their scale to higher energies—this should be studied in detail elsewhere. They become massless as the Higgs is switched off, reflecting the non-chiral nature of the underlying $\mathcal{N} = 4$ theory.

The Higgs sector is found to be more intricate than in the standard model, consisting of two doublets, which form an intrinsic part of the internal branes. This should lead to some protection from quantum corrections. The electroweak scale is set by the geometrical scale of the internal compact branes. Another important parameter is the rank N of the internal matrices, which determines the size of the “large” internal fuzzy spheres, and in particular the hierarchy of the Yukawa couplings for the standard model fermions versus the mirror fermions. The standard model Yukawas can be made

arbitrarily small for $N \rightarrow \infty$, while those of the mirror sector remain fixed at tree level. However, to assess the viability of the resulting model, quantum corrections due to the towers of massive modes must be taken into account.

There are many open questions and issues raised by this work. One important issue is the scale of the mirror fermions. A reliable computation of this and other physical parameters requires computing the quantum corrections due to integrating out the Kaluza–Klein tower of massive fields. Due to the rich spectrum this is a formidable task even at one loop, which should be feasible, however. Understanding the quantum contributions to the effective potential is also essential to clarify the stability of the background, and to clarify whether it is necessary to add a stabilizing term such as (32) by hand. Ultimately, this should also allow the selection of preferred backgrounds among the mini-landscape of matrix model configurations.

Although we have intentionally hidden any non-commutative aspects, it should be clear that our solution really defines a fully non-commutative version of the (extended) standard model on quantized Minkowski space \mathbb{R}_θ^4 . If the required stabilizing potential indeed arises through quantum effects in $\mathcal{N} = 4$ SYM, the model can be expected to be perturbatively finite and free of pathological UV/IR mixing, in contrast to previous proposals [47–49]. Moreover, gravity is expected to be included automatically in the matrix model, encoded in the trace- $U(1)$ sector [8,9,50]. However, this is not yet fully understood.

Another open issue is the assumed non-linear stabilization of the singlet Higgs S . A related aspect is the possible Majorana mass term for ν_R , which is expected to arise due to S .

If the above issues can be resolved in a satisfactory way, many interesting physical issues could be addressed, including in particular the physical properties of the Higgs. In any case, we have certainly demonstrated that there is no fundamental obstacle to obtaining near-standard-model physics from the matrix model.

Acknowledgements

This work is supported by the Austrian Science Fund (FWF): P24713. Useful discussions with J. Nishimura and A. Tsuchiya are gratefully acknowledged.

Funding

Open Access funding: SCOAP³.

Appendix A. Clifford algebra and reduction to four dimensions

The ten-dimensional Clifford algebra, generated by Γ_A , naturally separates into a four-dimensional and a six-dimensional one as follows:

$$\begin{aligned}\Gamma_A &= (\Gamma_\mu, \Gamma_{3+a}), \\ \Gamma_\mu &= \gamma_\mu \otimes \mathbb{1}_8, \quad \Gamma_{3+a} = \gamma_5 \otimes \Delta_a.\end{aligned}\tag{A1}$$

Here, the γ_μ define the four-dimensional Clifford algebra, and are chosen to be real corresponding to the Majorana representation in four dimensions for $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. Then $\gamma_0 = -\gamma_0^\dagger = -\gamma_0^T$ and $\gamma_i = \gamma_i^\dagger = \gamma_i^T$. The Δ_a define the six-dimensional Euclidean Clifford algebra, and are chosen to be real and antisymmetric. The ten-dimensional chirality operator

$$\Gamma = \gamma_5 \otimes \Gamma^{(\text{int})}\tag{A2}$$

separates into four- and six-dimensional chirality operators

$$\begin{aligned}\gamma_5 &= -i\gamma_0 \dots \gamma_3 = \gamma_5^\dagger = -\gamma_5^T, \\ \Gamma^{(\text{int})} &= -i\Delta_1 \dots \Delta_6 = (\Gamma^{(\text{int})})^\dagger = -(\Gamma^{(\text{int})})^T.\end{aligned}\quad (\text{A3})$$

Let us denote the ten-dimensional charge conjugation operator as

$$\mathcal{C} = C^{(4)} \otimes C^{(6)}, \quad (\text{A4})$$

where $C^{(4)}$ is the four-dimensional charge conjugation operator and $C^{(6)} = \mathbb{1}_8$ in our conventions. This operator satisfies, as usual, the relation

$$\mathcal{C}\Gamma^M\mathcal{C}^{-1} = -(\Gamma^M)^T. \quad (\text{A5})$$

Then the Majorana condition in 9+1 dimensions is¹⁸ $\Psi^C = \mathcal{C}\bar{\Psi}^T = \Psi$, hence

$$\Psi^* = \Psi, \quad \bar{\Psi} = \Psi^T \mathcal{C}, \quad (\text{A6})$$

since $\mathcal{C} = C^{(4)} = \gamma_0$ in the Majorana representation with real γ_μ . Thus the spinor entries are Hermitian matrices in a MW basis. The fermionic action can then be written as

$$\text{Tr}\bar{\Psi}\Gamma_a[X^a, \Psi] = \text{Tr}\Psi^T\gamma_0(\not{D}_4 + \gamma_5\not{D}_{(\text{int})})\Psi, \quad (\text{A7})$$

where

$$\not{D}_{(\text{int})} = \sum_{a=4}^9 \Delta^a[X_a, \cdot], \quad \{\not{D}_{(\text{int})}, \Gamma^{(\text{int})}\} = 0 \quad (\text{A8})$$

denotes the Dirac operator on the internal space. The most general 32-component Dirac spinors Ψ satisfying the Weyl constraint $\Gamma\Psi = \Psi$ as well as the Majorana condition $\Psi^* = \Psi$ can then be written as

$$\Psi = \sum_{i=1}^4 \left(\chi_{L,i} \otimes \eta_{L,i} + (\chi_{L,i} \otimes \eta_{L,i})^* \right) \quad (\text{A9})$$

where $\chi_{L,i}$ are left-handed spinors of $\mathfrak{so}(3, 1)$, labeled by four left-handed Weyl spinors $\eta_{L,i}$ of $\mathfrak{so}(6) \cong \mathfrak{su}(4)$. In particular, for spinors taking values in some block matrix as above, the Majorana condition amounts to

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{12}^* & \psi_{22} \end{pmatrix}, \quad \psi_{ii}^* = \psi_{ii}. \quad (\text{A10})$$

This means that the lower-diagonal matrices ψ_{12}^* are nothing but “anti-particles” of the upper-diagonal “particles,” so that there is no further doubling of the chiral fermions ψ_{12} stretching from brane 1 to brane 2, as identified in the text. Spelling out this block-matrix structure of the fermions, the Yukawa couplings are

$$\begin{aligned}\text{Tr}\bar{\Psi}\gamma_5\not{D}_{\text{int}}\Psi &= \text{Tr}\left(\psi_{11}^T\gamma_0\gamma_5\Delta_c[X_{(1)}^c, \psi_{11}] + \psi_{22}^T\gamma_0\gamma_5\Delta_c[X_{(2)}^c, \psi_{22}] \right. \\ &\quad \left. + \psi_{12}^T\gamma_0\gamma_5\Delta_c(X_{(2)}^c\psi_{12}^* - \psi_{12}^*X_{(1)}^c) + \psi_{12}^{*T}\gamma_0\gamma_5\Delta_c(X_{(1)}^c\psi_{12} - \psi_{12}X_{(2)}^c)\right),\end{aligned}\quad (\text{A11})$$

where $X^c = \text{diag}(X_{(1)}^c, X_{(2)}^c)$. Two of these terms coincide:

$$\text{Tr}\psi_{12}^{*T}\gamma_0\gamma_5X_{(1)}^c\Delta_c\psi_{12} = -\text{Tr}X_{(1)}^c\psi_{12}^T\Delta_c\gamma_0\gamma_5\psi_{12}^*, \quad (\text{A12})$$

using the Grassmann nature of ψ , so that

$$\text{Tr}\bar{\Psi}\gamma_5\not{D}_{\text{int}}\Psi = \text{Tr}\left(\psi_{11}^T\gamma_0\gamma_5\not{D}_6\psi_{11} + \psi_{22}^T\gamma_0\gamma_5\not{D}_{\text{int}}\psi_{22} + 2\psi_{12}^{*T}\gamma_0\gamma_5\not{D}_{\text{int}}\psi_{12}\right). \quad (\text{A13})$$

This gives rise to the Yukawa couplings computed in Sect. 4.3.

¹⁸ Note that T transposes only the spinor.

References

- [1] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. B, **498**, 467 (1997) [[arXiv:hep-th/9612115](#)].
- [2] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D, **55**, 5112 (1997) [[arXiv:hep-th/9610043](#)].
- [3] B. de Wit, J. Hoppe, and H. Nicolai, Nucl. Phys. B, **305**, 545 (1988).
- [4] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, Nucl. Phys. B, **565**, 176 (2000) [[arXiv:hep-th/9908141](#)].
- [5] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, Prog. Theor. Phys. **99**, 713 (1998) [[arXiv:hep-th/9802085](#)].
- [6] W. Taylor, Rev. Mod. Phys. **73**, 419 (2001) [[arXiv:0101126](#) [hep-th]].
- [7] D. Kabat and W. I. Taylor, Phys. Lett. B, **426**, 297 (1998) [[arXiv:9712185](#) [hep-th]].
- [8] H. Steinacker, Class. Quant. Grav. **27**, 133001 (2010) [[arXiv:1003.4134](#) [hep-th]].
- [9] H. Steinacker, JHEP **0712**, 049 (2007) [[arXiv:0708.2426](#) [hep-th]].
- [10] S.-W. Kim, J. Nishimura, and A. Tsuchiya, Phys. Rev. Lett. **108**, 011601 (2012) [[arXiv:1108.1540](#) [hep-th]].
- [11] Y. Ito, S.-W. Kim, Y. Koizuka, J. Nishimura, and A. Tsuchiya, [arXiv:1312.5415](#) [hep-th].
- [12] A. Chatzistavarakidis, H. Steinacker, and G. Zoupanos, JHEP **1109**, 115 (2011) [[arXiv:1107.0265](#) [hep-th]].
- [13] A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. **18B**, 29 (1991).
- [14] I. Chepelev and A. A. Tseytlin, Nucl. Phys. B, **511**, 629 (1998) [[arXiv:hep-th/9705120](#)].
- [15] M. Alishahiha, Y. Oz, and M. M. Sheikh-Jabbari, JHEP **9911**, 007 (1999) [[arXiv:hep-th/9909215](#)].
- [16] Y. Kitazawa and S. Nagaoka, JHEP **0602**, 001 (2006) [[arXiv:hep-th/0512204](#)].
- [17] D. N. Blaschke and H. Steinacker, JHEP **1110**, 120 (2011) [[arXiv:1109.3097](#) [hep-th]].
- [18] J. Nishimura and A. Tsuchiya, JHEP **1312**, 002 (2013) [[arXiv:1305.5547](#) [hep-th]].
- [19] H. Steinacker and J. Zahn, SIGMA **9**, 067 (2013) [[arXiv:1309.0650](#) [hep-th]].
- [20] H. Aoki, Prog. Theor. Phys. **125**, 521 (2011) [[arXiv:1011.1015](#) [hep-th]].
- [21] M. Berkooz, M. R. Douglas, and R. G. Leigh, Nucl. Phys. B, **480**, 265 (1996) [[arXiv:hep-th/9606139](#)].
- [22] I. Antoniadis, E. Kiritsis, and T. N. Tomaras, Phys. Lett. B, **486**, 186 (2000) [[arXiv:hep-ph/0004214](#)].
- [23] G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan, and A. M. Uranga, JHEP **0102**, 047 (2001) [[arXiv:hep-ph/0011132](#)].
- [24] I. Antoniadis, E. Kiritsis, J. Rizos, and T. N. Tomaras, Nucl. Phys. B, **660**, 81 (2003) [[arXiv:hep-th/0210263](#)].
- [25] L. E. Ibanez, F. Marchesano, and R. Rabadan, JHEP **0111**, 002 (2001) [[arXiv:hep-th/0105155](#)].
- [26] C. Kokorelis, Nucl. Phys. B, **677**, 115 (2004) [[arXiv:hep-th/0207234](#)].
- [27] R. Blumenhagen, M. Cvetič, P. Langacker, and G. Shiu, Ann. Rev. Nucl. Part. Sci. **55**, 71 (2005) [[arXiv:hep-th/0502005](#)].
- [28] F. Marchesano, Fortsch. Phys. **55**, 491 (2007) [[arXiv:hep-th/0702094](#)].
- [29] J. Arnlind, J. Choe, and J. Hoppe, [arXiv:13010757](#) [math.QA].
- [30] J. Madore, Class. Quant. Grav. **9**, 69 (1992).
- [31] J. Hoppe, *Quantum theory of a massless relativistic surface and a two-dimensional bound state problem* (MIT, Cambridge, MA, 1982), PhD thesis.
- [32] J. Hoppe, [[arXiv:hep-th/0206192](#)].
- [33] H. Grosse, F. Lizzi, and H. Steinacker, Phys. Rev. D, **81**, 085034 (2010) [[arXiv:1001.2703](#) [hep-th]].
- [34] P. Aschieri, T. Grammatikopoulos, H. Steinacker, and G. Zoupanos, JHEP **0609**, 026 (2006) [[arXiv:hep-th/0606021](#)].
- [35] S. Morelli, [arXiv:0907.3877](#) [hep-ph].
- [36] C. Coriano, N. Irges, and E. Kiritsis, Nucl. Phys. B **746**, 77 (2006) [[arXiv:hep-ph/0510332](#)].
- [37] J. Preskill, Annals Phys. **210**, 323 (1991).
- [38] H. Steinacker, Prog. Theor. Phys. **126**, 613 (2012) [[arXiv:1106.6153](#) [hep-th]].
- [39] J. Hoppe, [arXiv:hep-th/9702169](#).
- [40] A. P. Polychronakos, H. Steinacker, and J. Zahn, Nucl. Phys. B, **875**, 566 (2013) [[arXiv:1302.3707](#) [hep-th]].
- [41] A. M. Perelomov, *Generalized Coherent States and their Applications* (Springer, Berlin, 1986).
- [42] H. Grosse and P. Presnajder, Lett. Math. Phys. **28**, 239 (1993).

- [43] D. Berenstein and E. Dzienkowski, Phys. Rev. D, **86**, 086001 (2012) [[arXiv:1204.2788](#) [hep-th]].
- [44] H. Steinacker, JHEP **0902**, 044 (2009) [[arXiv:0812.3761](#) [hep-th]].
- [45] D. N. Blaschke and H. Steinacker, Class. Quant. Grav. **27**, 235019 (2010) [[arXiv:1007.2729](#) [hep-th]].
- [46] M. Drewes, Int. J. Mod. Phys. E, **22**, 1330019 (2013) [[arXiv:1303.6912](#) [hep-ph]].
- [47] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt, Eur. Phys. J. C, **23**, 363 (2002) [[arXiv:hep-ph/0111115](#)].
- [48] V. V. Khoze and J. Levell, JHEP **0409**, 019 (2004) [[arXiv:hep-th/0406178](#)].
- [49] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Eur. Phys. J. C, **29**, 413 (2003) [[arXiv:hep-th/0107055](#)].
- [50] H. S. Yang, Mod. Phys. Lett. A, **25**, 2381 (2010) [[arXiv:1007.1795](#) [hep-th]].