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Fragments of Kripke–Platek set theory and the metamathematics of α-recursion theory

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Abstract The foundation scheme in set theory asserts that every nonempty class has an $\in$-minimal element. In this paper, we investigate the logical strength of the foundation principle in basic set theory and α-recursion theory. We take KP set theory without foundation (called KP^-) as the base theory. We show that $\mathrm{KP}^- + \Pi_1$-Foundation $+ V = L$ is enough to carry out finite injury arguments in α-recursion theory, proving both the Friedberg-Muchnik theorem and the Sacks splitting theorem in this theory. In addition, we compare the strengths of some fragments of KP.

Keywords Metamathematics · Foundation · Kripke–Platek set theory · α-recursion theory

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1 Introduction

Denote by KP^- the theory obtained from the usual Kripke–Platek set theory KP by taking away the foundation scheme. By *fragments of* KP we mean subtheories of KP that include KP^-. These fragments arise naturally in the metamathematics of α-recursion theory, where one investigates the amount of foundation needed to prove various theorems in this subject.

Some research has been done on fragments of KP. An investigation of the logical strength of fragments of KP can be found in Ressayre's notes [21], where he showed that the hierarchy of Σ_n-foundation schemes is strict (cf. Theorem 4.15 below). Studying recursion-theoretic properties in fragments of KP is also called γ-recursion by Lubarsky [13]. He provided a study of models of KP^- in which Π_n-foundation fails and ω_1^{CK} represents the least non-recursive r.e. degree in the sense of the model. In Lubarsky's paper, Friedman's solution [7] to Post's problem for β-recursion was adopted to prove a splitting theorem. The proof-theoretic and set-theoretic aspects of fragments of KP were investigated by Cantini [3,4] and Rathjen [18–20]. Cantini [4] studied KP_1^-, i.e. $\mathrm{KP}^- +$ infinity $+\ \Pi_1$-foundation, and identified the smallest Σ-model for it. Here, a Σ*-model* for KP_1^- is some level L_α of the constructible hierarchy which satisfies all Σ_1 formulas provable from KP_1^-. Rathjen gave [19] a proof-theoretic analysis of primitive recursive set functions in the axiom system of $\mathrm{KP}^- +$ infinity $+\ \Pi_1$-foundation (which he called Σ_1-foundation in his papers), and characterized the logical strength of $\mathrm{KP}^- +$ infinity $+\ \Sigma_{n+2}$-foundation by the smallest ordinal α such that L_α is a model of all Π_2 sentences provable in the theory [18].

The metamathematics of α-recursion theory is partly motivated by the research in reverse recursion theory, and more generally, the metamathematics of classical recursion theory. In reverse recursion theory, we have models of arithmetic with limited *induction*, the analogue of foundation in arithmetic. Paris and Kirby [17] showed that Σ_{n+1}-induction ($\mathrm{I}\Sigma_{n+1}$) is strictly stronger than Σ_{n+1}-bounding ($\mathrm{B}\Sigma_{n+1}$), and that Σ_{n+1}-bounding is strictly stronger than Σ_n-induction.[1] In fact, $\mathrm{B}\Sigma_{n+1}$ is equivalent to $\mathrm{I}\Delta_{n+1}$ modulo the totality of exponentiation, as shown by Slaman [26]. The metamathematics of classical recursion theory was started by Simpson, who observed that $\mathrm{I}\Sigma_1$ is sufficient to prove the Friedberg–Muchnik theorem. Then Mytilinaios [15] showed that $\mathrm{I}\Sigma_1$ is enough for finite injury ($\mathbf{0}'$-priority) arguments, and Mytilinaios and Slaman [16] showed that in $\mathrm{I}\Sigma_2$, one can carry out infinite injury ($\mathbf{0}''$-priority) arguments. Although the original proof of the Sacks Density theorem seems to involve more than infinite injury, Groszek et al. [8] showed that surprisingly, $\mathrm{B}\Sigma_2$ is sufficient for the Density theorem.

α-recursion theory studies the computational properties of the *admissible* L_α's, i.e., those that satisfy KP. Sacks and Simpson [22] showed that the Friedberg–Muchnik theorem is valid in every admissible L_α. The Splitting and Density theorems were established by Shore [23,24]; they hold in every admissible L_α. The existence of a minimal pair is a typical example of an infinite injury argument in classical recursion

[1] $\mathrm{B}\Sigma_{n+1}$ essentially says that the Σ_{n+1}-formulas are closed under bounded quantification.

theory. Yet, whether it is true in every admissible L_α is open. See the papers by Lerman and Sacks [12], Maass [14] and Shore [25] for some partial results.

α-recursion theory has influenced the metamathematics of classical recursion theory. A subset of L_α is said to be *regular* if its intersection with any α-finite set is α-finite (where the α-finite sets are precisely the elements of L_α). The notion of regular sets originated from Sacks and Simpson [22], and Shore [23]. A *cut* is an example of non-regular set. It is known [5] that the degree of a cut can be a minimal degree, and it can also form a minimal pair with some $\emptyset^{(n)}$. Shore's blocking method [23,24] was introduced to solve the Splitting and Density problems in admissible L_α's. The Splitting problem in reverse recursion theory was solved using a similar method [15] in $\mathrm{I}\Sigma_1$.

There is much overlap between the techniques and results of α-recursion theory and the metamathematics of classical recursion theory. The reason for having such overlap is yet to be found. The research in this paper involves nonstandard models of set theory. These models are "between" those in nonstandard arithmetic and those in α-recursion theory. It is an initial attempt to understand the mysterious connections between these two areas.

The structure of the paper is as follows: Sect. 2 lists some basic definitions, axioms and propositions that are useful later. Sect. 3 applies these propositions to the Schröder–Bernstein theorem and shows that this theorem is provable in Π_1-Foundation. In Sect. 4 we discuss the L-hierarchy in models of fragments of KP and apply this hierarchy to separate Π_n-Foundation and Σ_n-Foundation. And Sects. 5 and 6 are devoted to the Friedberg–Muchnik theorem and the Splitting theorem respectively and prove they hold in any model of $\mathrm{KP}^- + \Pi_1$-Foundation $+ V = L$.

2 Preliminaries

2.1 Fragments of KP

Kripke–Platek set theory (KP) consists of the Extensionality, Foundation, Pairing and Union axioms together with Σ_0-Separation and Σ_0-Collection:

(i) Extensionality: $\forall x, y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$.
(ii) Foundation: If y is not a free variable in $\phi(x)$, then $[\exists x \phi(x) \to \exists x(\phi(x) \wedge \forall y \in x \neg\phi(y))]$.
(iii) Pairing: $\forall x, y \exists z(x \in z \wedge y \in z)$.
(iv) Union: $\forall x \exists y \forall z \in x \forall u \in z(u \in y)$.
(v) Σ_0-Separation: $\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \phi(z)))$ for each Σ_0 formula ϕ.
(vi) Σ_0-Collection: $\forall x[(\forall y \in x \exists z \phi(y, z)) \to \exists u \forall y \in x \exists z \in u \phi(y, z)]$ for each Σ_0 formula ϕ.

Here, Σ_0 formulas have only bounded quantifiers.

KP does not contain the Infinity axiom. If it is necessary for our theorems, then we will state the Infinity axiom explicitly.

(vii) Infinity: $\exists x[\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)]$.

Foundation is the dual of Induction.

(viii) Induction: If y is not a free variable in $\phi(x)$, then $[\forall x(\forall y \in x\, \phi(y)) \rightarrow \phi(x)] \rightarrow (\forall x \phi(x))$.

Clearly, for every class Γ of formulas, Γ-Induction holds if and only if $\neg\Gamma$-Foundation holds, where $\neg\Gamma = \{\neg\phi : \phi \in \Gamma\}$.

We use KP^- to denote KP without Foundation (i.e. Clauses (i), (iii)–(vi)). By *fragments* of KP, we mean systems obtained from KP by restricting the foundation scheme.

Proposition 2.1 KP^- *proves the following:*

(1) Strong Pairing: $\forall x, y \exists z\, (z = \{x, y\})$.
(2) Strong Union: $\forall x \exists y\, (y = \bigcup x)$.
(3) Δ_1-Separation *and* Σ_1-Collection.
(4) Strong Σ_1-Collection*: Suppose f is a Σ_1 function. If* $\mathsf{dom}(f)$ *is a set, then* $\mathsf{ran}(f)$ *and* $\mathsf{graph}(f)$ *are sets.*
(5) Ordered Pair: $\forall x, y \exists z\, (z = (x, y))$.
(6) Cartesian Product: $\forall x, y \exists z\, (z = x \times y)$.

Proof The usual proofs [2, Sects. I.3 and I.4] work in KP^-. □

2.2 The Lévy hierarchy

In Proposition 2.1, Δ_1 and Σ_1 are as defined in the Lévy Hierarchy. In the Lévy Hierarchy, we usually consider *normalized* formulas, that is, formulas in the form of $Q_0 v_0 \ldots Q_{n-1} v_{n-1} \varphi$, where (a) $Q_0, \ldots, Q_{n-1}$ are alternating quantifiers, (b) $v_0 \ldots v_{n-1}$ are variables, and (c) φ is Δ_0, or equivalently, φ has only bounded quantifiers.

The Collection principle says that normalized formulas are closed under bounded quantification. Without full collection, say in KP or KP^-, such closure properties may be lost. This problem is more related to Collection than to Foundation.

Definition 2.2 We define the $*$-hierarchy of formulas here. Suppose $m \leq n$ are natural numbers.

$$
\begin{array}{cccc}
\Sigma_0^* = \Pi_0^* = \Sigma_0 & \neg\Sigma_n^* \subseteq \Pi_n^* & \neg\Pi_n^* \subseteq \Sigma_n^* & \\
\Sigma_n^* \wedge \Sigma_m^* \subseteq \Sigma_n^* & \Pi_n^* \wedge \Pi_m^* \subseteq \Pi_n^* & \Sigma_{n+1}^* \wedge \Pi_m^* \subseteq \Sigma_{n+1}^* & \Pi_{n+1}^* \wedge \Sigma_m^* \subseteq \Pi_{n+1}^* \\
(\exists x \in y \Sigma_n^*) \subseteq \Sigma_n^* & (\exists x \in y \Pi_n^*) \subseteq \Pi_n^* & (\forall x \in y \Sigma_n^*) \subseteq \Sigma_n^* & (\forall x \in y \Pi_n^*) \subseteq \Pi_n^* \\
(\exists x \Sigma_n^*) \subseteq \Sigma_n^* & (\exists x \Pi_n^*) \subseteq \Sigma_{n+1}^* & (\forall x \Sigma_n^*) \subseteq \Pi_{n+1}^* & (\forall x \Pi_n^*) \subseteq \Pi_n^*
\end{array}
$$

A Σ_n^* (Π_n^*, resp.) formula is *normalizable* if it is equivalent to a Σ_n (Π_n, resp.) formula. KP^- proves that Σ_1^* formulas are normalizable. However, even assuming KP, there may still be a Σ_2^* formula that is not normalizable.

Proposition 2.3 (KP^-) *Suppose ϕ and ψ are normalized formulas. Then*

(1) $\neg\phi$, $\phi \wedge \psi$ and $\phi \vee \psi$ are normalizable.
(2) If ϕ is Σ_n (Π_n, resp.), then $\exists x\, \phi$ and $\exists x \in y\, \phi$ ($\forall x\, \phi$ and $\forall x \in y\, \phi$, resp.) are normalizable.

Proposition 2.4 KP$^-$ + Σ_n-Collection $\vdash$ *for any* Σ_m *(*Π_m*, resp.) formula* ϕ*,* $m \le n$*,* $\forall x \in y\phi$ *(*$\exists x \in y\phi$*, resp.) is normalizable.*

Proof For $m = 0$, it is straightforward. Now suppose $n \ge m > 0$, and the statement is true for $m - 1$. Also, suppose u is a new variable and ϕ is in the form of $\exists v\psi$ ($\forall v\psi$, resp.), where ψ is normalized Π_{m-1} (Σ_{m-1}, resp.). Then $\forall x \in y\phi \equiv \forall x \in y\exists v\psi \equiv \exists u\forall x \in y\exists v \in u\psi$ ($\exists x \in y\phi \equiv \exists x \in y\forall v\psi \equiv \forall u\exists x \in y\forall v \in u\psi$, resp.), by Σ_n-Collection. Since $\exists v \in u\psi$ ($\forall v \in u\psi$, resp.) is Π_{m-1} (Σ_{m-1}, resp.) normalizable, $\forall x \in y\phi$ ($\exists x \in y\phi$, resp.) is normalizable. □

Corollary 2.5 KP$^-$ + Σ_n-Collection $\vdash$ *all* Σ_n^* *and* Π_n^* *formulas are normalizable. In particular, assuming* KP$^-$*, every* Σ_1^* *or* Π_1^* *formula is respectively equivalent to a* Σ_1 *or* Π_1 *formula.*

3 Transfinite induction and the Schröder–Bernstein theorem

In this section, we move to the semantic aspects of fragments of KP. From now on, we always assume $M \models$ KP$^-$. And if $x \in M$, then we say x is M*-finite*.

Definition 3.1 $\alpha \in M$ is an *ordinal* if α is transitive and linearly ordered by $\in$. An ordinal of the form $\alpha \cup \{\alpha\}$, where α is an ordinal, is a *successor*. An ordinal λ is *limit* if it is nonempty and not a successor. If α is zero or a successor and no $\beta \in \alpha$ is limit, then α is *finite*.

Note that an ordinal in M must be M-finite but it may not be finite. We use Ord^M to denote the class of ordinals in M and use $<$ to denote $\in$ on the ordinals. With Σ_0-Foundation, it is possible to develop the basic properties of ordinals.

Proposition 3.2 (KP$^-$ + Σ_0-Foundation)

(1) $0 = \emptyset$ *is an ordinal.*
(2) If α *is an ordinal, then* $\beta \in \alpha$ *is an ordinal and* $\alpha + 1 = \alpha \cup \{\alpha\}$ *is an ordinal.*
(3) $<$ *is a linear order on the ordinals.*
(4) For every ordinal α*,* $\alpha = \{\beta : \beta < \alpha\}$*.*
(5) If C *is a nonempty set of ordinals, then* $\bigcap C$ *and* $\bigcup C$ *are ordinals,* $\bigcap C = \inf C = \mu\alpha(\alpha \in C)$ *and* $\bigcup C = \sup C = \mu\alpha(\forall\beta \in C(\beta \le \alpha))$*.*

Proof See Jech [9, Chapter 2] for the usual proofs. They go through in KP$^-$, as the reader can verify. □

Lemma 3.3 *If* $M \models$ Infinity*, then* M *has a limit ordinal. If* $M \models \Sigma_0$-Foundation *in addition, then* M *has a least limit ordinal* ω^M*.*

Proof Suppose $x \in M$ is a set witnessing the Infinity axiom. Let $C = \{\alpha \in x : \alpha$ is an ordinal$\}$. Then $\lambda = \sup C$ is an ordinal by Proposition 3.2, so that for any $\beta < \lambda$, there is an $\alpha \in x$ such that $\beta \le \alpha$. Since $\alpha + 2 = \alpha \cup \{\alpha\} \cup \{\alpha \cup \{\alpha\}\} \in C$, $\beta + 1 < \alpha + 2 \le \lambda$. Hence, λ is limit. □

Theorem 3.4 (Transfinite Induction along the ordinals) *Suppose $M \models \Pi_1$-Foundation and $I : M \to M$ is a Σ_1 partial function. Then the partial function $f : \mathsf{Ord}^M \to M, \delta \mapsto I(f \restriction \delta)$ is well defined and Σ_1. Moreover, if for all ordinals δ and all M-finite functions $\eta : \delta \to M$, we have $\eta \in \mathsf{dom}(I)$, then f is total.*

Proof f is Σ_1 definable:

$$f(\delta) = x \leftrightarrow \exists w\, (w \text{ is a function with domain } \delta \cup \{\delta\} \text{ such that } \forall \delta' \leq \delta\, [w(\delta') = I(w \restriction \delta') \wedge w(\delta) = x]).$$

Firstly, note that Σ_1 definable set $\mathsf{dom}(f)$ is downward closed and so by Π_1-Foundation, it is either Ord^M or an ordinal in M. Suppose $\delta \in \mathsf{Ord}^M$ and $x, x' \in M$ such that $f(\delta) = x \neq x' = f(\delta)$. Then we pick witnesses w for $f(\delta) = x$ and w' for $f(\delta) = w'$. By comparing w and w', we find the least $\delta' \leq \delta$ such that $w(\delta') \neq w'(\delta)$. However, this contradicts the fact that $w \restriction \delta' = w' \restriction \delta'$. Hence, f is a function.

Now we suppose that $\mathsf{dom}(f)$ is not Ord^M but for all ordinals δ and all M-finite functions $\eta : \delta \to M$, we have $\eta \in \mathsf{dom}(I)$. Pick the least ordinal $\delta \notin \mathsf{dom}(f)$. Then $\forall \delta' < \delta \exists x' (f(\delta') = x')$. By Proposition 2.1, $\mathsf{graph}(f)$ exists. Thus, $f(\delta)$ is also defined. This is a contradiction. □

In most popular proofs of the Schröder–Bernstein theorem, for example, that in Jech [9, Theorem 3.2], we obtain the required bijection by an induction on ω. Such proofs normally go through in $\mathrm{KP}^- + \Pi_1$-Foundation+Infinity. Without the Axiom of Infinity, the proof breaks down because ω, although still Δ_0-definable, can no longer be used to bound quantifiers. Therefore, although the Schröder–Bernstein theorem is provable in $\mathrm{KP}^- + \Pi_1$-Foundation alone, apparently a separate argument is needed when Infinity fails.

We reduce the ¬Infinity case to arithmetic, in which the situation is well-known. The key to this reduction is a Σ_1-definable bijection between the universe and the ordinals, defined by $\in$-induction. As observed in Kaye and Wong [10], this requires the existence of *transitive closures*. Recall the *transitive closure* of a set x, denoted by $\mathrm{TC}(x)$, is the smallest transitive set that includes x.

Lemma 3.5 $\mathrm{KP}^- + \Pi_1\text{-Foundation} \vdash \forall x\, \exists y\, \mathrm{TC}(x) = y$.

Proof Follow Lemma 5.3 and Proposition 5.4 in Kaye and Wong [10].

Theorem 3.6 (Transfinite $\in$-induction) *Let $M \models \mathrm{KP}^- + \Pi_1$-Foundation, and $I : M \to M$ that is Σ_1-definable. Then there exists a Σ_1-definable $f : M \to M$ satisfying $f(x) = I(f \restriction x)$ for every $x \in M$.*

Proof Similar to that of Theorem 3.4. Transitive closures are used to show that such an f is total. □

(The inverse of) the following bijection between the universe and the ordinals originates from Ackermann [1].

Theorem 3.7 *Let* $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \neg$Infinity. *Then*

$$f(x) = \sum_{y \in x} 2^{f(y)}$$

defines a bijection $f : M \to \mathsf{Ord}^M$ *with a* Σ_1 *graph.*

Proof A standard application of $\in$-induction shows the functionality and totality of f. The failure of the Infinity Axiom contributes to the injectivity of f. If $\alpha \in \mathsf{Ord}^M$, then $f(\mathrm{Ack}(\alpha)) = \alpha$, where

$$\mathrm{Ack}(\alpha) = \{\mathrm{Ack}(\beta) : \exists \gamma{<}\alpha\ \exists \delta{<}2^\beta\ \alpha = (2\gamma + 1)2^\beta + \delta\},$$

defined by induction on the ordinals. □

In a sense, this theorem shows that $\neg$Infinity is a strong assumption over $\mathrm{KP}^- + \Pi_1$-Foundation, because it implies the Power Set Axiom, Π_1-Separation, the Axiom of Choice, and V = L. The Schröder–Bernstein theorem also follows as promised.

Theorem 3.8 ($\mathrm{KP}^- + \Pi_1$-Foundation) *Let* A, B *be sets. If there are injections* $A \to B$ *and* $B \to A$, *then there is a bijection* $A \to B$.

Proof We already mentioned that most standard proofs go through in $\mathrm{KP}^- + \Pi_1$-Foundation + Infinity. So suppose $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \neg$Infinity, and $f : M \to \mathsf{Ord}^M$ is the bijection given by Theorem 3.7. Take $A, B \in M$. Suppose M contains injections $A \to B$ and $B \to A$.

With Π_1-Foundation in M, we know $\mathsf{Ord}^M \models \mathrm{I}\Pi_1$ as a model of arithmetic. Via f, we may view A and B as (arithmetically) coded subsets of Ord^M. Apply $\mathrm{I}\Delta_0 + \exp$ in Ord^M to find $\alpha, \beta \in \mathsf{Ord}^M$ that are respectively bijective with A and B in M. The hypotheses imply that there are injections $\alpha \to \beta$ and $\beta \to \alpha$ coded in Ord^M. So by the coded version of the Pigeonhole Principle, which is available in all models of $\mathrm{I}\Delta_0$, we conclude $\alpha = \beta$. It follows that A is bijective with B. □

4 The constructible universe

4.1 Basic properties

In this section, M always satisfies $\mathrm{KP}^- + \Pi_1$-Foundation. By a transfinite induction, we may define L^M along Ord^M:

$$\begin{aligned} L_0^M &= \emptyset, \\ L_{\alpha+1}^M &= L_\alpha^M \cup \mathrm{Def}^M(L_\alpha^M), \\ L_\lambda^M &= \textstyle\bigcup_{\alpha<\lambda} L_\alpha^M \quad \text{where } \lambda \text{ is limit.} \end{aligned}$$

Here, $\mathrm{Def}^M(x)$ denotes the collection of all definable subsets of x in the sense of M. Let $L^M = \bigcup_{\alpha \in \mathsf{Ord}^M} L_\alpha^M$.

If Infinity holds, then we may define the function Def^M as usual. If Infinity fails, then we get Def^M using the power set axiom and Π_1-Separation given by Theorem 3.7.

Lemma 4.1 ($\mathrm{KP}^- + \Pi_1$-Foundation) *The predicate "$x \models \phi[\mathbf{v}/\mathbf{a}]$", where x is a set, ϕ is a formula (in the sense of the model) and* **a** *is a sequence of sets, is Δ_1. We denote this relation by* $\mathsf{Sat}(\ulcorner\phi\urcorner, x, \mathbf{a})$.

Proof (Sketch) The predicate "$x \models \phi[\mathbf{v}/\mathbf{a}]$" is true if and only if we have an M-finite function s which assigns to each triple $(\ulcorner\phi'\urcorner, x', \mathbf{a}')$ a truth value according to the usual definition of truth, and $s(\ulcorner\phi\urcorner, x, \mathbf{a})$ is assigned "true". With this definition, we cannot get conflicting truth assignments. This is proved by applying Σ_0-Foundation to the s's above. Also, every triple gets a truth value, because if not, then by Π_1-Foundation, there is a formula of minimum length such that this fails, which is not possible. □

Theorem 4.2 ($\mathrm{KP}^- + \Pi_1$-Foundation) *For every ordinal α, $L^M_\alpha \in M$. The function $\alpha \mapsto L^M_\alpha$ is Δ_1.*

Proposition 4.3 ($\mathrm{KP}^- + \Pi_1$-Foundation) *For every ordinal α, L^M_α is transitive and $L^M_\alpha \cap \mathsf{Ord}^M = \alpha$.*

Theorem 4.4 *If $M \models \mathrm{KP}^- + \Pi_1$-Foundation, then $L^M \models \mathrm{KP}^- + \Pi_1$-Foundation.*

Proof We only need to check Π_1-Foundation and Σ_0-Collection. Pick any Π_1 formula $\forall w \phi(x, w)$, where $\phi \in \Delta_0$. Suppose there is an $x \in L^M$ such that $\forall w \in L^M \phi(x, w)$. The set $\{y \in L^M_\alpha : \forall w \in L^M \phi(y, w)\}$ is Π_1 for every ordinal α. By Π_1-Foundation in M, it is either empty or has a $\in$-least witness. Hence, L^M satisfies Π_1-Foundation.

To check Δ_0-Collection, we fix any Δ_0-formula $\psi(y, w)$ with parameters from L^M and $x \in L^M$. Suppose $\forall y \in x \exists w \in L^M\ \psi(y, w)$. Then we may check the L-rank of the witnesses (i.e. the least α such that $w \in L^M_\alpha$). Then $\forall y \in x \exists \alpha \exists w \in L^M_\alpha\ \psi(y, w)$. By Σ_1-Collection of M, there is a searching bound $\alpha^* \in M$ such that $\forall y \in x \exists \alpha < \alpha^* \exists w \in L^M_\alpha\ \psi(y, w)$. Therefore, $L^M_{\alpha^*}$ is the searching bound for the witness w for all $y \in x$. □

Definition 4.5 ($\mathrm{KP}^- + \Pi_1$-Foundation) $V = L$ stands for $\forall x \exists \alpha (x \in L_\alpha)$.

If $M \models \mathrm{KP}^- + \Pi_1$-Foundation, then $L^M \models V = L$.

Lemma 4.6 *Suppose $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V} = \mathrm{L}$. Then there exists a Δ_1 bijection $M \to \mathsf{Ord}^M$ that preserves the relation $\in$.*

Corollary 4.7 *Let $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V} = \mathrm{L}$. Then there exists a Δ_1-definable linear order $<_\mathrm{L}$ on M such that M satisfies*

- $\forall s \left(\exists x\ (x \in s) \to \exists x \left(x \in s \wedge \forall x' {<_\mathrm{L}} x\ (x' \notin s)\right)\right)$;
- $\forall x, y\ (x \in y \to x <_\mathrm{L} y)$; *and*
- $\forall \alpha {\in} \mathsf{Ord}\ \forall x {\in} \mathrm{L}_\alpha\ \forall v {<_\mathrm{L}} x\ (v \in \mathrm{L}_\alpha)$.

Note that ω^M may not be the standard ω. Nevertheless, the notion of Σ_n formulas, where n is a standard positive natural number, is absolute, in the sense that the formulas recognized in M as Σ_n are all equivalent in M to some standard Σ_n formulas. That is because we have a universal Σ_1 formula that is standard, so that in a nonstandard Σ_n formula, we may code its Δ_0 matrix into a standard Σ_1 formula, if n is odd; and we may code it into a Π_1 formula, if n is even.

Lemma 4.8 ($\mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V} = \mathrm{L}$) *There are*

(a) a universal Σ_1 formula;
(b) a universal Σ_1 function, i.e., a recursive enumeration of all Σ_1 partial functions, and
(c) a universal Turing functional, i.e., a recursive enumeration of all the codes for oracle computations.

Proof Note that there is an effective enumeration of all Σ_1 formulas (in the sense of the model). The universal Σ_1 formula is just the one searching for a witness for each Σ_1 formula. The relation that "a Σ_1 formula φ is satisfied with a witness w" can be formalized by Lemma 4.1 (cf. Sect. 3.1 in Barwise [2]). For a universal Σ_1 function, given an index, we enumerate ordered pairs (x, y) such that no (x, y') has appeared earlier. One can define a universal Turing functional similarly. □

However, this is not the full picture of formulas within M. We can have a Σ_n formula for a nonstandard natural number $n \in M$. Also, we have limited collection. Thus, it is possible that we have a Σ_n^* formula, where $n \in \omega$ is standard, that is not equivalent to a Σ_n formula.

4.2 Recursive Ordinals in models of KP$^-$

Lemma 4.9 *If* $M \models \mathrm{KP}^- + \Pi_1$-Foundation + Infinity, *and* $\omega^M = \omega$, *then every recursive ordinal is in* M.

Proof For the sake of contradiction, consider the least recursive ordinal not in M and suppose Φ_k, $k < \omega$ codes a well ordering of ω isomorphic to this recursive ordinal. Then Φ_k together with its ordering is M-finite. Let $<_{\Phi_k}$ denote this ordering. Define a Σ_1 function $f : \omega \to \mathsf{Ord}^M$ as follows:

$$f(n) = \gamma \leftrightarrow$$
$$\text{there is an order isomorphism between } \gamma \text{ and } \{m < \omega : m <_{\Phi_k} n\}.$$

If f is total, then the order type of Φ_k is in M, leading to a contradiction. Otherwise, suppose n is $<_{\Phi_k}$-least such that $n \notin \mathsf{dom}(f)$. Since $\mathsf{dom}(f) = \{m < \omega : m <_{\Phi_k} n\}$ is M-finite, $\mathsf{graph}(f)$ is M-finite. It follows that f is an isomorphism between $\mathsf{dom}(f)$ and $\mathsf{ran}(f)$, contradicting with $n \notin \mathsf{dom}(f)$. □

We may generalize recursive ordinals as in the following.

Definition 4.10 ($\mathrm{KP}^- + \Pi_1$-Foundation + Infinity) An ordinal α is *recursive* if there is a Σ_1 (in the language of arithmetic with parameters in ω^M) linear ordering of ω^M with respect to which ω^M is order isomorphic to α.

Remark 4.11 Suppose there is an ordinal that is nonrecursive; then there must be a least one by Π_1-Foundation. We denote this ordinal by $\omega_1^{\mathsf{CK},M}$. In this case, it is straightforward to check that $L^M_{\omega_1^{\mathsf{CK},M}}$ satisfies full foundation. Furthermore, $L^M_{\omega_1^{\mathsf{CK},M}} \models \Delta_0$-Collection, and so $L^M_{\omega_1^{\mathsf{CK},M}} \models$ KP. To see this, for the sake of contradiction, assume Δ_0-Collection fails. Then there is a $\Delta_0(L^M_{\omega_1^{\mathsf{CK},M}})$ function f^* from ω^M cofinally to $\omega_1^{\mathsf{CK},M}$. For each ordinal δ in $\omega_1^{\mathsf{CK},M}$, we may pick up the least index e_δ such that Φ_e codes a linear ordering and the order type of Φ_{e_δ} is δ (From here onwards in this paragraph, each Φ_e is taken as a code of a binary relation "$<$", which may and may not be a linear ordering). Thus, f^* induces a function $f : n \mapsto e_{f^*(n)}$, for all $n \in \omega^M$. This function is $\Delta^1_1(M)$. Note that $\{f^*(n) : n \in \omega^M\}$ is cofinal in $\omega_1^{\mathsf{CK},M}$ and so is {order type of $\Phi_{f(n)} : n \in \omega^M$}. This implies that the set of Gödel numbers of well-orderings $\mathrm{WOG}(M) = \{e \in \omega^M : \Phi_e$ codes a well-ordering$\} = \{e \in \omega^M : \Phi_e$ has an order preserving map to $\Phi_{f(n)}$, for some $n\}$ is $\Sigma^1_1(M)$. Here, the second definition of $\mathrm{WOG}(M)$ is equivalent to the first because (1) if Φ_e is well-ordered, then the order type of Φ_e is less than $\omega_1^{\mathsf{CK},M}$, and (2) {order type of $\Phi_{f(n)} : n \in \omega^M$} is cofinal in $\omega_1^{\mathsf{CK},M}$. Kleene's representation theorem [11] indicates that every $\Pi^1_1(M)$ set is many-one reducible to $\mathrm{WOG}(M)$. Thus, the above equalities concerning $\mathrm{WOG}(M)$ imply that every $\Pi^1_1(M)$ set is $\Sigma^1_1(M)$, deriving a contradiction.

Remark 4.12 If every ordinal in Ord^M is recursive, then we write $\omega_1^{\mathsf{CK},M} = \mathsf{Ord}^M$. There is a model $M \models \mathrm{KP}^- + \Pi_1$-Foundation in which full foundation fails and $\omega_1^{\mathsf{CK},K} = \mathsf{Ord}^K$. To see this, let K be an ω-nonstandard elementary extension of the standard $\mathrm{L}_{\omega_1^{\mathsf{CK}}}$. Pick a nonstandard $c \in \omega^K$ and a large enough $n \in \mathbb{N}$ such that $\Sigma_{n+1} \supseteq \mathrm{KP}^- + \Pi_1\text{-Foundation} + \mathrm{V} = \mathrm{L}_{\omega_1^{\mathsf{CK}}}$. Let M be the substructure of K consisting of the Σ_{n+1}-definable elements over c. Then M is ω-nonstandard because $c \in \omega^M$. Also $M \preceq_{n+1} K$ by Tarski–Vaught, and so $M \models \mathrm{KP}^- + \Pi_1\text{-Foundation} + \mathrm{V} = \mathrm{L}_{\omega_1^{\mathsf{CK}}}$. However, the standard $\mathbb{N}$ is Σ_{n+3}-definable in M, because it is the set of all $b \in \omega^M$ such that some element of M is not definable over c by a Σ_{n+1}-formula with Gödel number less than b. So $M \not\models \Pi_{n+3}$-Foundation.

4.3 Collection, separation and foundation

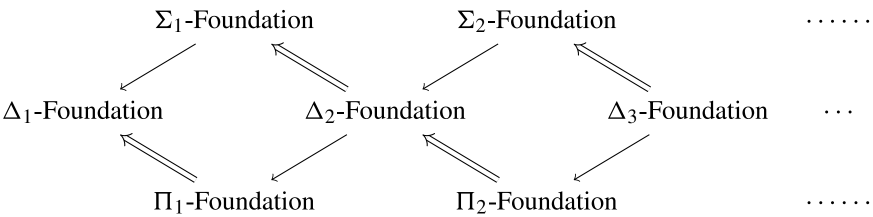
Collection, Separation and Foundation are closely related to each other. An immediate observation is that $\mathrm{KP}^- + \Gamma$-Separation $+\Sigma_0$-Foundation, together with the existence of transitive closures, implies Γ-Foundation. A less obvious result is the following:

Lemma 4.13 (KP^-) *For every standard natural number n,* Σ_n-Collection $\vdash$ Δ_n-Separation.

Proof We prove this by induction on n. Suppose we have proved the conclusion for n and Σ_{n+1}-Collection holds. Assume $\exists y \phi(x, y)$ and $\exists y \psi(x, y)$ are formulas such that (1) ϕ and ψ are Π_n, (2) $\forall x \in z\, \exists y\, (\phi(x, y) \vee \psi(x, y))$, and (3) $\neg\exists x \in z\, \exists y\, (\phi(x, y) \wedge \psi(x, y))$. Then Σ_{n+1}-Collection implies that there is a b such that $\forall x \in$

$z\ \exists y \in b\,(\phi(x, y) \vee \psi(x, y))$. By Δ_n-Separation, $z' = \{x \in z : \exists y \in b\,(\phi(x, y))\}$ and $z'' = z \setminus z'$ are sets. (Here, we need $\Pi_n^* = \Pi_n$, which is implied by Σ_{n+1}-Collection.) This shows Δ_{n+1}-Separation. □

Over KP^-, the following implications hold.



We will use the L hierarchy to show the implications indicated by double arrows above do not reverse.

Lemma 4.14 (Ramón Pino [21, Theorem 1.28]) *Let* $n \in \mathbb{N}$. *Then* $\mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Pi_{n+1}\text{-Foundation} + \mathrm{V} = \mathrm{L}$ *proves the following statement.*

For every $\delta \in \mathsf{Ord}$, *there exists a sequence* $(\alpha_i)_{i \leq \delta}$ *in which* $\alpha_0 = 0$ *and* $\alpha_{i+1} = \min\{\alpha > \alpha_i : \mathrm{L}_\alpha \preceq_n \mathrm{L}\}$ *for each* $i < \delta$.

Proof If $n = 0$, then the sequence we want is just $(\alpha)_{\alpha \leq \delta}$. So suppose $n > 0$. With Σ_1-Induction on ω, we have a Π_n-formula $\Pi_n\text{-}\mathsf{Sat}$ for the satisfaction of Π_n-formulas. We can find arbitrarily high levels of the L-hierarchy which reflect this formula thanks to Σ_{n+1}-Collection and Σ_{n+1}-Induction on ω. This implies there are arbitrarily large $\mathrm{L}_\alpha \preceq_n \mathrm{L}$. With Σ_{n+1}-Collection and Π_{n+1}-Foundation, we can iterate this along any ordinal. □

Theorem 4.15 (Ressayre [21, Theorem 4.6]) $\mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Sigma_{n+1}\text{-Foundation} + \mathrm{V} = \mathrm{L} \nvdash \Pi_{n+1}\text{-Foundation}$ *for all* $n \in \mathbb{N}$.

Proof Start with a countable $M \models \mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Pi_{n+1}\text{-Foundation} + \mathrm{V} = \mathrm{L}$ in which $\omega^M = \omega$ but Ord^M is not well-ordered. Take a nonstandard $\delta \in \mathsf{Ord}^M$. Let $(\alpha_i)_{i \leq \delta + \delta}$ be a sequence of ordinals given by Lemma 4.14. As δ is nonstandard, the reader can easily verify using a standard argument [6, Section 3] that there are continuum-many initial segments of Ord^M between δ and $\delta + \delta$. So at least one of them is not definable in M. Take any initial segment $I \subseteq \mathsf{Ord}^M$ with this property. We will prove that $K = \bigcup_{i \in I} \mathrm{L}_{\alpha_i}^M$ is the model we want.

Claim 4.15.1 $K \preceq_n M$.

Proof of claim We show by induction on $m \leq n$ that $K \preceq_m M$. Clearly $K \preceq_0 M$ because K is a transitive substructure of M. Let $m < n$ such that $K \preceq_m M$. Pick any $\phi(\bar{x}, \bar{z}) \in \Sigma_m$ and $\bar{c} \in K$. Assume $K \models \forall \bar{x}\ \phi(\bar{x}, \bar{c})$. Find some $i \in I$ such that $\bar{c} \in \mathrm{L}_{\alpha_i}^M$. Let $\bar{x} \in \mathrm{L}_{\alpha_i}^M$ be arbitrary. Then $K \models \phi(\bar{x}, \bar{c})$. Since $K \preceq_m M$ by the induction hypothesis, and $\mathrm{L}_{\alpha_i}^M \preceq_n M$, we know $\mathrm{L}_{\alpha_i}^M \models \phi(\bar{x}, \bar{c})$. Hence $\mathrm{L}_{\alpha_i}^M \models \forall \bar{x}\ \phi(\bar{x}, \bar{c})$. This transfers up to M by n-elementarity, completing the induction.

Claim 4.15.2 $K \models \Sigma_{n+1}$-Collection.

Proof of claim Take $a, \bar{c} \in K$ and $\phi(x, y, \bar{z}) \in \Pi_n$ such that

$$K \models \forall x{\in}a\ \exists y\ \phi(x, y, \bar{c}).$$

Pick any $j \in \delta + \delta$ above I. Let $x \in a$. Then $K \models \phi(x, y, \bar{c})$ for some $y \in K$. Since $K \preceq_n M$ and $\mathrm{L}^M_{\alpha_j} \preceq_n M$, the same is true when the satisfaction of ϕ is evaluated in $\mathrm{L}^M_{\alpha_j}$ instead. Therefore, by setting $b = \mathrm{L}^M_\alpha$ for some $\alpha < \alpha_j$ above I, we see that $\mathrm{L}^M_{\alpha_j} \models \exists b\ \forall x{\in}a\ \exists y{\in}b\ \phi(x, y, \bar{c})$.

Since the choice of $j \in \delta + \delta$ above I was arbitrary, this underspills. Let $i \in I$ and $b \in \mathrm{L}^M_{\alpha_i}$ such that $a, \bar{c} \in \mathrm{L}^M_{\alpha_i}$ and $\mathrm{L}^M_{\alpha_i} \models \forall x{\in}a\ \exists y{\in}b\ \phi(x, y, \bar{c})$. Notice since $\mathrm{L}^M_{\alpha_i} \preceq_n M$ and $K \preceq_n M$, we have $\mathrm{L}^M_{\alpha_i} \preceq_n K$. Therefore $K \models \forall x{\in}a\ \exists y{\in}b\ \phi(x, y, \bar{c})$ too because $\mathrm{L}^M_{\alpha_i}$ is a transitive substructure of K.

This claim implies $K \models \Delta_{n+1}$-Separation + Δ_{n+1}-Foundation.

Notice if $n = 0$, then we do not have Π_1-Foundation in K. Thus Corollary 4.7 does not always apply to K. Nevertheless, the model M does satisfy Π_1-Foundation, and so K can still get the conclusions of Corollary 4.7 from M.

Claim 4.15.3 $K \models \Sigma_{n+1}$-Foundation.

Proof of claim Let $\theta(v, x)$ be a Π_n-formula that may contain undisplayed parameters from K. Suppose

$$K \models \exists x\ \exists v\ \theta(v, x) \wedge \forall x\ \big(\exists v\ \theta(v, x) \rightarrow \exists x'{\in}x\ \exists v\ \theta(v, x')\big).$$

Fix any $x_0 \in K$ such that $K \models \exists v\ \theta(v, x_0)$. Let $\eta(k, x)$ be the formula

$$\begin{aligned} &(x)_0 = x_0 \\ &\wedge \forall i{\in}k \left(\begin{array}{l} (x)_{i+1} \in (x)_i \wedge \exists \alpha{\in}\mathsf{Ord}\ \exists v{\in}\mathrm{L}_\alpha \\ \left(\begin{array}{l} (x)_i \in \mathrm{L}_\alpha \wedge \theta(v, (x)_{i+1}) \\ \wedge \forall v'{<_\mathrm{L}}v\ \forall x'{\in}(x)_i\ \neg\theta(v', x') \\ \wedge \forall x'{<_\mathrm{L}}(x)_{i+1}\ \big(x' \in (x)_i \rightarrow \neg\theta(v, x')\big) \end{array} \right) \end{array} \right), \end{aligned}$$

which is Σ_{n+1} over M by Corollary 4.7 and Claim 4.15.2.

We show $K \models \forall k{\in}\omega\ \exists x\ \eta(k, x)$ by an external induction on k. Suppose we already have $x_0, x_1, \ldots, x_k \in K$ satisfying the inductive conditions. Take any large enough $\alpha \in \mathsf{Ord}^K$ such that $x_k \in \mathrm{L}^K_\alpha$ and $K \models \exists x{\in}x_k\ \exists v{\in}\mathrm{L}_\alpha\ \theta(v, x)$. Then we can set

$$\begin{aligned} v_{k+1} &= \min\nolimits_{<_\mathrm{L}}\{v \in \mathrm{L}^K_\alpha : K \models \exists x{\in}x_k\ \theta(v, x)\}, \text{ and} \\ x_{k+1} &= \min\nolimits_{<_\mathrm{L}}\{x \in x_k : K \models \theta(v_{k+1}, x)\}. \end{aligned}$$

These minima exist by Δ_{n+1}-Foundation.

Apply Σ_{n+1}-Collection to get $s \in K$ such that $K \models \forall k{\in}\omega\ \exists x{\in}s\ \eta(k, x)$. Define $f(k) = y$ to be

$$\exists x{\in}s\ \big(\eta(k, x) \wedge (x)_k = y\big),$$

which is Σ_{n+1} over M by Σ_{n+1}-Collection. It is not hard to verify that $K \models \forall k{\in}\omega\ \exists! y\ f(k) = y$. So the set

$$\{y \in \mathrm{TC}(x_0) : K \models \exists k{\in}\omega\ f(k) = y\}$$

is Δ_{n+1}-definable but has no $\in$-minimum element. This contradicts Δ_{n+1}-Foundation in K.

Notice that $K \models \mathrm{V}{=}\mathrm{L}$ because the L-hierarchies in M and K, being Δ_1-definable, coincide.

Claim 4.15.4 $K \not\models \Pi_{n+1}$-Foundation.

Proof of claim If $n = 0$, then K contains δ but not $\delta + \delta$, so that Π_1-Foundation fails in K. Suppose $n > 0$. Then $\delta{+}\delta \in K$ by Π_1-Foundation, but there can be no sequence $(\beta_i)_{i \leq \delta+\delta}$ in which $\beta_0 = 0$ and $\beta_{i+1} = \min\{\beta > \beta_i : \mathrm{L}_\beta \preceq_n \mathrm{L}\}$ for each $i < \delta + \delta$, because $K \preceq_n M$. So Lemma 4.14 tells us K cannot satisfy Π_{n+1}-Foundation. □

In particular, this theorem says that if $n \in \mathbb{N}$, then $\mathrm{KP}^- + \Sigma_{n+1}$-Foundation $\nvdash$ Π_{n+1}-Foundation. We do not see how to show this without invoking the much stronger Σ_{n+1}-Collection. The use of the Infinity Axiom is necessary, because $\mathrm{KP}^- + \Sigma_{n+1}$-Foundation $+ \neg$Infinity $\vdash \Pi_{n+1}$-Foundation, as is classically known in the context of arithmetic [17]. The use of $\mathrm{V}{=}\mathrm{L}$, however, is only superficial: we may as well work with L^M if $M \models \mathrm{V}{\neq}\mathrm{L}$ in the proof above. Also, we may repeat the proof of Theorem 4.15 with $V = L$ replaced by $V = L[R]$ for some real R.

Compare the next theorem with Proposition 3.2 in Rathjen [19].

Theorem 4.16 $\mathrm{KP}^- + \Sigma_{n+1}$-Collection $+ \Pi_{n+1}$-Foundation $+ \mathrm{V}{=}\mathrm{L} \vdash \Sigma_{n+1}$-Foundation *for all* $n \in \mathbb{N}$.

Proof If Infinity holds, then the proof is the same as that of Claim 4.15.2, except that now, we can use Π_{n+1}-Foundation to show $\forall k{\in}\omega\ \exists x\ \eta(k, x)$. If Infinity fails, then apply the equivalence between $\mathrm{I}\Pi_{n+1}$ and $\mathrm{I}\Sigma_{n+1}$ in arithmetic [17] via the bijection given by Theorem 3.7. □

Question 4.17 *Let* $n \in \mathbb{N}$. *Does* KP^-, *Infinity,* Σ_{n+1}*-Collection, plus* Π_{n+1}*-Foundation prove* Σ_{n+1}*-Foundation?*

4.4 Level 1-KPL

Definition 4.18 *Level 1*-KPL denotes $\mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V}{=}\mathrm{L}$.

Notice Theorem 4.16 above implies Level 1-KPL $\vdash \Sigma_1$-Foundation.

Definition 4.19 (*Level 1-KPL*) Let I be a bounded initial segment of ordinals. We say that I is a *cut*, if there is no least ordinal $\beta \notin I$.

Note in the above definition, though I is transitive and linearly ordered by $\in$, I is not an ordinal, as otherwise, I would become the least ordinal not in I.

Lemma 4.20 (Level 1-KPL) *For all $n \geq 1$, Σ_n-*Foundation *holds if and only if there is no Π_n cut. The same is true for Π_n-Foundation and Σ_n cuts if Σ_n-Collection is additionally assumed.*

Proof If there is a Π_n cut, then Σ_n-Foundation fails, clearly. Conversely, suppose Π_n-Induction fails. That is, there is a Π_n formula $\phi(x)$ such that $\forall x[(\forall y \in x\phi(y)) \rightarrow \phi(x)]$ but for some x_0, $\neg\phi(x_0)$ holds. Let $f : \mathsf{Ord}^M \rightarrow M$ be the recursive bijection in Lemma 4.6. Then we check that $\forall\alpha \in \mathsf{Ord}^M[(\forall\beta < \alpha\phi(f(\beta))) \rightarrow \phi(f(\alpha))]$, as f preserves $\in$ of M. Now we define $I = \{\alpha \in \mathsf{Ord}^M : \forall\beta < \alpha\phi(f(\beta))\}$. Then I is bounded Π_n and there is no least ordinal not in I. Thus, I is a Π_n cut. □

Lemma 4.21 (Level 1-KPL) *Every M-finite set x has a cardinality $|x|$.*

Lemma 4.22 (Level 1-KPL). *If δ is an infinite cardinal, then there is an order preserving bijection from δ into δ^2, where $(a, b) \prec (c, d)$ if and only if* $\max(a, b) < \max(c, d) \vee (\max(a, b) = \max(c, d) \wedge a < c) \vee (\max(a, b) = \max(c, d) \wedge a = c \wedge b < d)$.

Proof For the sake of contradiction, we assume that δ is the least cardinal that fails to have this property. We define the function by Σ_1 induction along the ordinals. Note that the maximum of the two coordinates of the image of α is no more than α for any $\alpha < \delta$. Thus, the domain of the function has to be greater than δ. Let the image of δ be (a, b), where $\max(a, b) < \delta$. Considering the order preserving bijection from $|\max(a, b)|^2$ and $|\max(a, b)|$, we can get a surjection from $|\max(a, b)|$ onto δ. That is a contradiction. □

Corollary 4.23 (Level 1-KPL). *Suppose δ is an infinite cardinal. Then $|\delta^2| = \delta$. Thus, for x and y satisfying $|x|, |y| \leq \delta$, the Cartesian product $x \times y$ and the set $x^{<\omega}$ of finite sequences of x are both of cardinality at most δ. Thus, for every infinite ordinal α, $|L_\alpha| \leq |\alpha|$.*

Proof Let $|x| \leq \delta$. Consider the sequence $\{x_n\}_{n<\omega}$. Now, by Σ_1-Induction, $|x_n| \leq \delta$ for all $n < \omega$. Thus, $|x^{<\omega}| \leq |\delta \times \omega| \leq \delta$.

For the sake of contradiction, assume that α is the least infinite ordinal such that there is no injection from L_α into α. If α is a successor ordinal with predecessor α', then $|L_\alpha| \leq |L_{\alpha'}{}^{<\omega} \times \omega| \leq |\alpha' \times \omega| \leq |\alpha'|$, which contradicts our assumption. Thus, α is a limit ordinal. Since for any infinite $\beta < \alpha$, $|L_\beta| \leq |\beta| \leq |\alpha|$, there is a Σ_1 injection from L_α to $\alpha \times \alpha$. Thus, $|L_\alpha| \leq |\alpha|$, which again is a contradiction. □

5 The Friedberg–Muchnik theorem

In this section we will show the Friedberg–Muchnik Theorem in Level 1-KPL. Again M is a model of Level 1-KPL. The Sack–Simpson construction [22] in α-recursion theory uses the Σ_2-*cofinality* (of the ordinals), i.e., the least ordinal that can be mapped to a cofinal set of ordinals by a Σ_2 function, the existence of which apparently needs much more foundation than Level 1-KPL can afford.

Question 5.1 *Is there a model of Level 1-*KPL *with no Σ_n cofinality for some $n \geq 2$?*

Lemma 5.2 (Level 1-KPL) *If there is a Σ_1 injection from the universe into an ordinal, then there is the least such an ordinal. It is called the Σ_1* projectum*, denoted by $\sigma 1 p(M)$, or $\sigma 1 p$ for short.*

Proof Suppose $\alpha \in M$ is an ordinal such that there is a Σ_1 injection from the universe into α. We claim $|\alpha| = \sigma 1 p$. Clearly, there is a Σ_1 injection from the universe into $|\alpha|$. Conversely, if we have a Σ_1 injection p from M into $\beta \leq |\alpha|$, then $p \restriction |\alpha|$ is M-finite and is an injection into β. As $|\alpha|$ is a cardinal in M, $\beta = \alpha$. □

Similarly, we may define the Σ_2 *projectum* of M, $\sigma 2 p(M)$, to be the least ordinal such that there is a Σ_2 injection from the universe into it. However, it is not known whether such a projectum exists.

Question 5.3 *Is there a model of Level 1-*KPL *with no Σ_2 projectum?*

Corollary 5.4 (Level 1-KPL) *If $\sigma 1 p(M)$ exists, then $\sigma 1 p(M)$ is the largest cardinal in M.*

Proof Suppose $\sigma 1 p(M)$ exists and α is any ordinal in M greater than $\sigma 1 p(M)$. It is sufficient to show that $|\alpha| = \sigma 1 p(M)$. The proof of Lemma 5.2 tells us that $\sigma 1 p(M)$ is a cardinal. Moreover, since there is a Σ_1 injection from Ord^M into $\sigma 1 p(M)$, there is such an injection from Ord^M into α. Therefore, α satisfies the assumptions in the proof of Lemma 5.2. Hence, $|\alpha = \sigma 1 p(M)$. □

Lemma 5.5 (Level 1-KPL) *Given an r.e. set A, we have a recursive enumeration of A without repetition. I.e. there is a recursive one-one function f such that $\mathsf{dom}(f)$ is Ord^M or an ordinal in M, and $\mathsf{ran}(f) = A$.*

Proof Suppose at each stage s, only ordinals less than s are enumerated into A. That is, by any stage, only M-finitely many ordinals are enumerated into A. Then we define f by transfinite induction:

$f(\delta) = \lambda$ if and only if there is a stage $s$such that
(1) all ordinals enumerated before stage $s$are included in $f \restriction \delta$, and
(2)λ is the least ordinal enumerated at stage s but not in $f \restriction \delta$.

It is straightforward to check that f is the function we want. □

Corollary 5.6 (Level 1-KPL) *If $\sigma 1 p(M)$ exists, then every Σ_1 subset of an ordinal less than $\sigma 1 p(M)$ is M-finite. If $\sigma 1 p(M)$ does not exist, then every Σ_1 bounded subset of Ord^M is M-finite.*

Proof Let A be any Σ_1 subset as in the statement of Corollary 5.6. By Lemma 5.5, there is a recursive one-one function f such that $\mathsf{dom}(f)$ is Ord^M or an ordinal in M, and $\mathsf{ran}(f)$ is A. To show that A is M-finite, it is sufficient to prove that $\mathsf{dom}(f)$ is an ordinal. Now we suppose $\mathsf{dom}(f)$ is Ord^M, for the sake of contradiction.

Case 1. $\sigma 1p(M)$ exists. Then f is an injection from Ord^M to an ordinal less than $\sigma 1p(M)$.

Case 2. $\sigma 1p(M)$ does not exist. Then f is an injection from Ord^M to an ordinal in M.

In both cases, we get a contradiction. □

Definition 5.7 (*Level 1-KPL*) Suppose δ is an ordinal. We say δ is *(Σ_1) stable* if L_δ is a Σ_1 elementary substructure of the whole model.

Lemma 5.8 (Level 1-KPL) *For every γ such that $\omega \leq \gamma < \sigma 1p$, there is a stable ordinal $\delta \geq \gamma$ with the same cardinality as γ.*

Proof Let γ be an ordinal such that $\omega < \gamma < \sigma 1p$ and x be the set of all finite sequences of L_γ. Suppose $f: M \to \mathsf{Ord}^M$ is the bijection from Lemma 4.6 and $\{\varphi_e\}$ is a universal enumeration of all Σ_1 formulas as in Lemma 4.8.

Consider the set $y = \{(e, \mathbf{a}) : e \in \omega, \mathbf{a} \in x$, the number of free variables in φ_e is equal to the dimension of $\mathbf{a}$ plus one$\}$. Note that $|y| \leq |\omega \times |x|| \leq \gamma < \sigma 1p$. Thus, any Σ_1 subset of y is M-finite.

Now we define a (partial) map $g: y \to M$ such that $(e, \mathbf{a}) \mapsto$ the least v (in the order of f) such that $\varphi_e(v, \mathbf{a})$ holds. As $\mathsf{dom}(g)$ is M-finite, so is $\mathsf{ran}(g)$.

Let $G = \mathsf{ran}(g)$. Then $|G| \leq |y| \leq |\gamma|$. Note that $L_\gamma \subset G$. (Then $x, y \subset G$). Thus, $|G| = |\gamma|$. Suppose φ is a Σ_1 formula (in the sense of M), and $\mathbf{a}$ is a finite sequence in G such that the number of free variables in φ is equal to the dimension of $\mathbf{a}$ plus one and $M \models \exists v \varphi(v, \mathbf{a})$. We claim that $G \models \exists v \varphi(v, \mathbf{a})$. To see this, let $\boldsymbol{\phi}$ be an M-finite sequence of Σ_1 formulas with parameters from L_γ. Then M, and thus G, is a model of $\exists v \exists \mathbf{a}[\varphi(v, \mathbf{a})$ and each coordinate of $\mathbf{a}$ satisfies the corresponding coordinate in $\boldsymbol{\phi}]$. This yields that $G \prec_1 M$.

Now we define the Mostowski collapse c of G as follows:

$$\begin{aligned} c(v) = z \leftrightarrow \exists \eta(&\eta \text{ is a function such that} \\ &\forall v \in \mathsf{dom}(\eta)(\eta(v) = \{\eta(v') : v' \in v \cap G\}) \text{ and } \eta(v) = z) \end{aligned}$$

Note that c is Σ_1 definable and $\mathsf{dom}(c) = G$ by Π_1-Foundation. Let $G' = \mathsf{ran}(c)$, which is M-finite.

For every $v, v' \in G$, $v \in v' \leftrightarrow c(v) \in c(v')$ by Π_1-Foundation. Also, if $M \models v \neq v'$, then $M \models v \triangle v' \neq \emptyset$ and so $G' \models c(v) \neq c(v')$. Hence c is an isomorphism. Thus, for every ordinal in G, its image in G' is still an ordinal. Thus, $G' \subset \bigcup_{\alpha \in \mathsf{Ord}^{G'}} L_\alpha$. Conversely, $G' \supset \bigcup_{\alpha \in G'} L_\alpha$, since G' is transitive and $G \models \forall$ ordinal α, L_α exists. Let δ be the least ordinal not in G'. Then $G' = L_\delta$.

Consider the function g. Note that for every $(e, \mathbf{a}) \in \mathsf{dom}(g)$, $c((e, \mathbf{a})) = (e, \mathbf{a})$, and $g(e, \mathbf{a})$ is the least witness for $\varphi_e(v, \mathbf{a})$. Thus, the same is still true in G'. For this reason, $G' = G$. □

5.1 Construction

Now we are ready to construct r.e. subsets A and B of the ordinals such that $A \not\leq_T B$ and $B \not\leq_T A$ as claimed in the Friedberg–Muchnik Theorem. Here $A \not\leq_T B$ means A *is not setwise reducible to* B, i.e., there is no r.e. set Φ such that for any M-finite set F,

$$F \subseteq A \Leftrightarrow \exists P \subseteq B \exists N \subseteq \overline{B}(\langle F, 1, P, N\rangle \in \Phi)$$
$$F \subseteq \overline{A} \Leftrightarrow \exists P \subseteq B \exists N \subseteq \overline{B}(\langle F, 0, P, N\rangle \in \Phi),$$

where P, N are M-finite. Our construction will yield sets A, B that are not pointwise reducible to each other. Here we say A *is pointwise reducible to* B if there is an r.e. set Φ such that for any $x \in M$,

$$x \in A \Leftrightarrow \exists P \subseteq B \exists N \subseteq \overline{B}(\langle x, 1, P, N\rangle \in \Phi)$$
$$x \notin A \Leftrightarrow \exists P \subseteq B \exists N \subseteq \overline{B}(\langle x, 0, P, N\rangle \in \Phi).$$

It turns out that the construction we give in this subsection works well only in the case when there is no maximum cardinal or when $\sigma 1 p$ exists in the model; see Theorem 5.12 below. Later in Sect. 5.3, we will present a slightly modified construction which will deal with the other case.

Let $p\colon \mathsf{Ord}^M \to \sigma 1 p$ be a Σ_1 injection. (If $\sigma 1 p$ does not exist, then p is an arbitrary Σ_1 one-one function from Ord^M to Ord^M, e.g. p could be the identity function.) Let $\{\Phi_\epsilon : \epsilon \in \mathsf{Ord}^M\}$ be a uniform sequence of all Σ_1 Turing functionals (for pointwise reducibility). Requirements are either $\Phi_\epsilon^A \neq B$ or $\Phi_\epsilon^B \neq A$ for some ordinal ϵ. Let $\{R_\epsilon : \epsilon \in \mathsf{Ord}^M\}$ be a Σ_1 enumeration of all requirements. We say R_ϵ has *higher priority* than $R_{\epsilon'}$ if $p(\epsilon) < p(\epsilon')$. At any stage γ,

- R_ϵ *requires attention* if $\epsilon < \gamma$, R_ϵ was not satisfied prior to stage γ, and for the corresponding witness, Turing machine, and the oracle known so far, the outcome of the computation on this witness is 0 and this witness is not in the scope of any restrictions of any requirement seen by stage γ to be of higher priority;
- R_ϵ *receives attention* if
  (1) it requires attention;
  (2) we enumerate the witness into the corresponding set; and
  (3) we put restrictions on the usage of the computation;
- R_ϵ is *initialized* if we erase the memories of all activities of R_ϵ by stage γ and assign a new witness for it;
- R_ϵ is *satisfied* if it received attention at some previous stage, and after that until the present stage, it has not been initialized.

Suppose we are at stage $\gamma \in \mathsf{Ord}^M$. Consider $\{R_\epsilon : \epsilon < \gamma\}$. If there is a requirement requiring attention, then we satisfy the one with the highest priority seen at the current stage, say R_{ϵ_0} and initialize all requirements in $\{R_\epsilon : \epsilon < \gamma\}$ of lower priorities. If no requirement requires attention, then we initialize all R_ϵ (together with the lower-priority requirements) with $\epsilon < \gamma$ such that a new element of the range of p less than $p(\epsilon)$ is enumerated exactly at this stage. Then one by one, for each requirement in

$\{R_\epsilon : \epsilon < \gamma\}$ that has not been satisfied nor assigned a witness not in the scope of restrictions by higher priority requirements seen by stage γ, we assign a new witness for it. We do not need to assign a new witness for R_ϵ if either R_ϵ has been satisfied by stage γ or R_ϵ has a witness not in the scope of restrictions by higher priority requirements seen by stage γ.

Here is the method to assign new witnesses: We take the collection of all requirements in $\{R_\epsilon : \epsilon < \gamma\}$ that require witnesses. Suppose they are $\{R_{\epsilon_i} : i < \gamma'\}$ such that $\forall i' < i < \gamma'(\epsilon_{i'} < \epsilon_i)$. We assign witnesses by induction on i. Assume that each $R_{\epsilon_{i'}}$, $i' < i$ has gained a witness. Then we take w as the witness of R_{ϵ_i} if and only if w is the least ordinal such that w has not been a witness so far and it is not in the scope of restrictions by higher priority requirements seen at the current stage.

5.2 Verification

Lemma 5.9 (Level 1-KPL) *Successor infinite cardinals are regular.*

Proof Suppose δ is a successor cardinal, its predecessor cardinal is δ^- and $\{\alpha_i : i < \beta\}$ is an M-finite sequence of ordinals such that $\alpha_i, \beta < \delta$. Then there is a Σ_1, thus M-finite, bijection from $\{(x, i) : x \in \alpha_i, i < \beta\}$ into $(\delta^-)^2$. Thus, $|\bigcup\{\alpha_i : i < \beta\}| \leq \delta^-$. □

Lemma 5.10 (Level 1-KPL + Infinity) *Suppose*

(1) δ is a regular cardinal in M,
(2) α is an ordinal less than δ, and
(3) $\{X_i : i < \alpha\}$ is a uniform r.e. sequence of M-finite sets of ordinals with cardinality less than δ. That is, the set $\{\langle i, \beta\rangle : i < \alpha, \beta \in X_i\}$ is r.e., and for every $i < \alpha$, X_i is an M-finite subset of Ord^M, and for every $i < \alpha$, $|X_i| < \delta$.

Then $\bigcup_{i<\alpha} X_i$ is M-finite and $|\bigcup_{i<\alpha} X_i| < \delta$.

Proof The idea here originated from Sacks and Simpson's paper [22].

Without loss of generality, we assume that the X_i's are mutually disjoint. Because $\{\langle i, \beta\rangle : i < \alpha, \beta \in X_i\}$ is r.e., so is $\bigcup_{i<\alpha} X_i$. By Lemma 5.5, there is a recursive one-one function f such that $\mathsf{dom}(f)$ is either Ord^M or an ordinal in M, and $\mathsf{ran}(f) = \{\langle i, \beta\rangle : i < \alpha, \beta \in X_i\}$. To show the conclusion in Lemma 5.10 via a contradiction, we assume that $\mathsf{dom}(f)$ is either Ord^M or an ordinal not less than δ. Then $\mathsf{ran}(f \upharpoonright \delta)$ is M-finite. Let $\varphi(i, \beta, w)$ be a Δ_0 formula such that

$$\forall i < \alpha \forall \beta(\beta \in X_i \leftrightarrow \exists w\, \varphi(i, \beta, w)).$$

Then

$$\forall \beta \in \mathsf{ran}(f \upharpoonright \delta)\exists w\, \exists i < \alpha\, \varphi(i, \beta, w).$$

Σ_1-Collection shows that

$$\exists w^* \forall \beta \in \mathsf{ran}(f \upharpoonright \delta)\exists w \in w^* \exists i < \alpha\, \varphi(i, \beta, w).$$

Fix any $i < \alpha$. Note

$$X_i \cap \mathsf{ran}(f \restriction \delta) = \{\beta \in \mathsf{ran}(f \restriction \delta) : \exists w \in w^* \varphi(i, \beta, w)\}$$

is M-finite. Thus $f^{-1}(X_i \cap \mathsf{ran}(f \restriction \delta)) = f^{-1}(X_i) \cap \delta$ is also M-finite. Moreover,

$$|f^{-1}(X_i) \cap \delta| = |f^{-1}(X_i \cap \mathsf{ran}(f \restriction \delta))| = |X_i \cap \mathsf{ran}(f \restriction \delta)| \leq |X_i| < \delta.$$

Since δ is regular, $\sup(f^{-1}(X_i) \cap \delta) < \delta$. Note that

$$\bigcup_{i<\alpha}(f^{-1}(X_i) \cap \delta) = (\bigcup_{i<\alpha} f^{-1}(X_i)) \cap \delta = \delta.$$

Hence, $\{\sup(f^{-1}(X_i) \cap \delta) : i < \alpha\}$ is cofinal in δ, contradicting the regularity of δ. □

Lemma 5.11 (Level 1-KPL) *If there is no maximum cardinal, then the cardinals are cofinal in* Ord^M.

Proof By Lemma 5.4, $\sigma 1 p$ does not exist. Thus, Lemma 5.6 yields that every bounded r.e. set of ordinals is M-finite. For the sake of contradiction, suppose all cardinals are bounded by γ. Then the set $\{\alpha < \gamma : \alpha$ is not a cardinal$\}$ is a bounded r.e. set and so is M-finite. Thus, $C = \{\alpha < \gamma : \alpha$ is a cardinal$\}$ is M-finite as well. Let δ be the least ordinal not in C. Then $\delta \notin C$, but it is a cardinal. □

Theorem 5.12 *(Level 1-*KPL*) If there is no maximum cardinal or* $\sigma 1 p$ *is in the model, then all requirements in the construction are satisfied.*

Proof If at some stage R_ϵ is satisfied and never initialized afterwards, then we are done. Otherwise, let γ be a stage at which all elements in $\mathsf{ran}(p) \restriction p(\epsilon)$ have been enumerated. Note that if there is no maximum cardinal, then $\sigma 1 p$ does not exist and p is an arbitrary Σ_1 function from Ord^M to Ord^M.

Let $\{S_j\}$ be the enumeration of the requirements with higher priorities than R_ϵ and R_ϵ itself with priority ordering. Then this sequence is M-finite and of length less than a regular cardinal δ in the model. Now let $I_j = \{\alpha \leq$ order type of stages at which S_j is initialized or assigned a new witness$\}$. Then the sequence $\{I_j\}$ is uniformly enumerable. We claim that each I_j is M-finite and less than δ. Otherwise, let j be the least such that $I_j \supseteq \delta$. By Lemma 5.10, $\bigcup_{j'<j} I_{j'}$ is M-finite and less than δ. Let ξ be the least stage such that $\bigcup_{j'<j} I_{j'}$ has been enumerated completely. Then by stage ξ, I_j cannot be more than the order type of $1 + 2 \times \bigcup_{j'<j} I_{j'}$. After stage ξ, I_j is initialized at most once. Thus, I_j is no more than the order type of $1+2\times\bigcup_{j'<j} I_{j'}+3$, not containing δ as a subset.

Thus, after some stage γ', R_ϵ is never initialized nor assigned a new witnesses. If R_ϵ requires attention, then it would be the one with highest priority and is satisfied and never injured afterwards. Otherwise, the witness would show that R_ϵ is satisfied automatically. □

5.3 Modified construction and its verification

Now we consider the case that $\sigma 1 p$ is not in the model and there is the maximal cardinal. We denote the maximum cardinal by $\aleph$.

The set $\{\delta > \aleph : \delta$ is not stable$\}$ is an r.e. set, and so, by Corollary 5.6, it is regular. Recall that a collection of ordinals in M is regular, if its intersection with any ordinal in M is M-finite. At each stage s, we say that δ is *stable at stage* s if $\aleph < \delta < \aleph + 1 + s$ and according to the information up to $\aleph + 1 + s$, we think that δ is stable. Then $\delta > \aleph$ is stable if and only if there is a stage s such that for all stages $t \geq s$, δ is stable at stage t. In fact, by Corollary 5.6 and Σ_1-Collection, for any $\alpha > \aleph$, there is a stage s such that after stage s, our justification of the stability of any ordinal in $(\aleph, \alpha]$ will never change.

At stage s, let $\delta_1^s < \delta_2^s < \cdots < \delta_i^s < \cdots$ be an enumeration of all stable-at-stage-s ordinals greater than $\aleph$. Let $\delta_0^s = 0$. $[\delta_i^s, \delta_{i+1}^s)$ is called *block* i *at stage* s. Then for every ordinal α, there is a stage s such that after stage s, all blocks below α will not be changed. For every block i at stage s, let h_i^s be the least (in the order of L) M-finite injection from block i at stage s into $\aleph$. If $\delta_1^s < \cdots < \delta_{i+1}^s$ are not changed from stage s onwards, then so are block i and h_i^s.

We do the construction of A and B as in Sect. 5.1 with the following priority order:

R_ϵ has *higher priority* than $R_{\epsilon'}$ if there are a stage s and blocks $i \leq j$ which are not changed from stage s onwards, such that

(1) ϵ is in block i and ϵ' is in block j, and
(2) either $i < j$, or $i = j$ and $h_i^s(\epsilon) < h_i^s(\epsilon')$.

This priority order is not recursive. Yet, for every ordinal α, the priority order on the set $\{R_\epsilon : \epsilon < \alpha\}$ can be recursively approximated and from some stage onwards, the approximation gives a correct order on $\{R_\epsilon : \epsilon < \alpha\}$. At each stage, we do the construction via the approximation of the priority order.

Other parts of the construction are parallel to that in Sect. 5.1. The rest of this section will give a detailed description. Readers familiar with this can skip to the verification, i.e., Lemma 5.13.

At stage s, we say that

- R_ϵ *requires attention* if
  (1) the least stable ordinal δ at stage s such that $\epsilon < \delta < s$ exists;
  (2) there is a stage $t < s$ such that $\{\alpha \leq \delta : \alpha$ is stable at stage $t\} = \{\alpha \leq \delta : \alpha$ is stable at stage $s\}$; and
  (3) R_ϵ was not satisfied prior to stage s and for the corresponding witness, Turing machine, and the oracle known so far, the outcome of the computation on this witness is 0 and that witness is not in the scope of any restrictions of higher-priority (according to our knowledge at stage s) requirements;
- R_ϵ *receives attention* if
  (1) it requires attention;
  (2) we enumerate the witness into the corresponding set; and
  (2) we put restrictions on the usage of the computation;
- R_ϵ is *initialized* if we erase the memories of all activities of R_ϵ by stage s and assign a new witness for it;

- R_ϵ is *satisfied* if it received attention at some previous stage, and after that until the present stage, it has not been initialized.

Suppose we are at stage $s \in \mathsf{Ord}^M$. Consider $\{R_\epsilon : \epsilon < s\}$. If there is a requirement requiring attention, then we satisfy the one with the highest priority, say R_ϵ and initialize all requirements in $\{R_\epsilon : \epsilon < s\}$ of lower priorities. If no requirement requires attention, then we initialize all R_ϵ (together with the lower-priority requirements) with $\epsilon < s$, such that its block has been changed at this stage or its map into $\aleph$ is changed at this stage, i.e. no $t, \delta < s$ satisfy

(i) $\epsilon < \delta < t$;
(ii) δ is stable at stage s (and so stable at stage t);
(iii) $\{\alpha \leq \delta : \alpha \text{ is stable at stage } t\} = \{\alpha \leq \delta : \alpha \text{ is stable at stage } s\}$; and
(iv) if ϵ is in block i at stage t, then for every $t' \in [t, s]$, $h_i^{t'} = h_i^t$.

Lastly, one by one, for each requirement in $\{R_\epsilon : \epsilon < s\}$ that has not been satisfied nor assigned a witness not in the scope of restrictions by higher priority requirements seen at current stage, we assign a new witness for it.

The following lemma implies that every requirement is satisfied eventually. The difficulty is to show that for every requirement, requirements of higher priority only M-finitely many times take "actions", including receiving attention, being initialized and being assigned with new witnesses. If the requirements of higher priority stop actions from some stage onwards, then the requirement being considered will have a chance to be satisfied eventually. Lemma 5.13 below overcomes the difficulty using stable ordinals: each requirement stops actions before the second next stable ordinal.

Lemma 5.13 (Level 1-KPL) *Fix an i such that $\delta_j = \lim_s \delta_j^s$ exists for every $j \leq i+2$. We denote $\lim_s \delta_j^s$ by δ_j, $j \leq i+2$. Let $I_\epsilon = \{s : R_\epsilon$ receives attention, is assigned a new witness, or is initialized at stage $s\}$. If $\epsilon \in [\delta_i, \delta_{i+1})$, then $I_\epsilon \in L_{\delta_{i+2}}$.*

Proof The stability of δ_{i+1} implies $\{\delta \leq \delta_{i+1}^s : \delta \text{ is stable at stage } s\} = \{\delta_j : j \leq i+1\}$ for all $s > \delta_{i+1}$. To show Lemma 5.13, it suffices to show that for every $j \leq i$, if the requirement R_{ϵ_j} is in block j, then $I_{\epsilon_j} \in L_{\delta_{j+2}}$. Suppose not. Let j be least which witnesses this, and $s_0 > \delta_{j+1}$ be least such that $h^{s_0} = h_j$ is found. Then $s_0 < \delta_{j+2}$.

By the definition of h_j^s, from stage s_0 onwards, all requirements in blocks $< j$ will not receive attention nor be initialized. For every R_ϵ in block j, we consider the set $I'_\epsilon = \{\alpha : \text{the order type of } I_\epsilon \setminus s_0 \text{ is no less than } \alpha\}$. Let $\delta \leq \aleph$ be any infinite regular cardinal. If for every R_ϵ in block j such that the priority order of R_ϵ, restricted to block j, is less than δ, we can get $I'_\epsilon < \delta$, then we are done. Otherwise, let R_ϵ be the one with the highest priority in block j such that $I'_\epsilon \geq \delta$. Then $U = \bigcup\{I_{\epsilon'} \setminus s_0 : R_{\epsilon'} \text{ is in block } j \text{ and has higher priority than } R_\epsilon\}$ is a union of fewer than δ many M-finite sets, each of cardinality less than δ. By Lemma 5.10, U is M-finite with cardinality less than δ. Thus, $\eta = \sup\{I'_{\epsilon'} : R_{\epsilon'} \text{ is in block } j \text{ and } R_{\epsilon'} \text{ has higher priority than } R_\epsilon\} < \delta$, and so $I'_\epsilon \leq 3 \times \eta + 2 < \delta$. That is a contradiction.

6 The splitting theorem and the blocking method

In this section, we prove the Sacks Splitting theorem in the setting of Level 1-KPL. We fix a regular nonrecursive r.e. set X and we will split X into two r.e. sets A and B such that

(1) $A \cup B = X$,
(2) $A \cap B = \emptyset$,
(3) $X \not\leq_T A$, and
(4) $X \not\leq_T B$.

To satisfy (1) and (2), we enumerate the elements in X one by one and put them into either A or B but not both. For (3) and (4), we deal with the requirements

$$P_e : \Phi_e^A \neq X$$
$$Q_e : \Phi_e^B \neq X$$

for all $e \in \mathsf{Ord}^M$.

For a single requirement, we apply the classical method of preserving computation. To settle all requirements, we adopt the blocking method as in α-recursion theory. The problem is that, within Level 1-KPL, we may not have the Σ_2 cofinality of the Ord^M. Thus, here we use a modified version that came from arithmetic [15]. It is a modified version of that in α-recursion theory. Here, a block is determined by its previous actions: we only stop enlarging a block when the actions of all its previous blocks terminate. The next lemma says that each block either grows to infinity or reaches to a limit at some M-finite stage.

Lemma 6.1 *For any nondecreasing recursive sequence $\{\xi_s\}_s$, either it is cofinal in Ord^M (we denote this by $\lim_s \xi_s = \infty$) or there is a stage s such that for all $t > s$, $\xi_t = \xi_s$.*

Proof Suppose $\{\xi_s\}_s$ is bounded in Ord^M. Let δ be the least ordinal such that for all s, $\xi_s \leq \delta$. We note that $\forall \delta' < \delta \exists s(\xi_s > \delta')$. Then Σ_1-Collection tells us there is a stage s_0 such that $\forall \delta' < \delta \exists s < s_0(\xi_s > \delta')$. Thus, $\xi_{s_0} = \delta$ and we are done.

6.1 Construction

Now we construct A and B stage by stage. We may pick an enumeration of X such that at each stage s, there is at most one element enumerated into X and that element (if any) is less than s. The set of elements enumerated into X before stage s is denoted by $X_{<s}$. Similarly, we use $A_{<s}$, $B_{<s}$, etc.

We say a requirement is a *P-requirement* (a *Q-requirement*, resp.) or of *P-type* (*Q-type*, resp.), if it is of the form $\Phi_e^A \neq X$ ($\Phi_e^B \neq X$, resp.). One essential principle in the blocking method is that there is only one type of requirements in any block.

Block α at stage s is $[0, h(\alpha, s))$, where $h(\alpha, s) =$

- 1, if $\alpha = 0$; (In the rest of the definition of h, we do not consider the case $\alpha = 0$.)

- $\alpha + 1$, if $s = 0$;
- some value δ to be specified in the construction such that $\delta \geq h(\alpha, t)$ for all $t < s$ and $\delta > h(\beta, s)$ for all $\beta < \alpha$, if $\alpha, s > 0$.

We say α is *even* if $\alpha = \gamma + 2n$ for some limit ordinal γ and some finite ordinal n. Otherwise, α is *odd*. We always assign P-requirements to even blocks and Q-requirements to odd blocks. More precisely, for instance, suppose α is even and *stable up to stage* s, i.e., there is $t < s$ such that for all stages $t' \in [t, s)$ and all $\beta \leq \alpha$, $h(\beta, t') = h(\beta, t)$. Then let *the* α*th requirement at stage* s, which we denote by R^s_α, be $\bigwedge\{P_\lambda : \lambda \text{ is in Block } \alpha \text{ at stage } s\}$. For ordinals in odd blocks, Q-requirements are assigned similarly.

Also, we define the *maximum common length of* R^s_α, denoted by $\mathbf{m}(\alpha, s)$, as follows: If there is a stage $t < s$ such that $h(\alpha, t) \geq s$, then let $\mathbf{m}(\alpha, s) = 0$. Otherwise, suppose α is even and e is in Block α up to stage s.[2] Then

$$m(e, s) = \sup\{l < s : \Phi_e^{A_{<s}} \upharpoonright l = X_{<s} \upharpoonright l\}.$$

Correspondingly, the *reservation of* R^s_e, denoted by $r(e, s)$, is the least ordinal $r \leq s$ such that, $\Phi_e^{A_{<s} \upharpoonright r} \upharpoonright m(e, s) = X_{<s} \upharpoonright m(e, s)$. Similarly, define $m(e, s)$ and $r(e, s)$ using B instead of A when e is in an odd block up to stage s. For every block α, let $\mathbf{r}(\alpha, s) = \sup\{r(e, s) : e \text{ is in Block } \alpha \text{ up to stage } s\}$, and $\mathbf{m}(\alpha, s) = \sup\{m(e, s) : e \text{ is in Block } \alpha \text{ up to stage } s\}$.

At stage $s > 0$, let R^s_α, $\mathbf{m}(\alpha, s)$, $\mathbf{r}(\alpha, s)$ be defined as above. If no element is enumerated into X, then let $A_s = A_{<s}$, $B_s = B_{<s}$ and $h(\alpha, s) = \max\{\sup_{t<s} h(\alpha, t), \sup_{\beta<\alpha}(h(\beta, s) + 1)\}$ for all α, s.

Now suppose x is enumerated into X at stage s. Let $\alpha \leq s$ be the least such that $x < \mathbf{r}(\alpha, s)$. If no such α exists, then enumerate x into A and $h(\alpha, s) = \max\{\sup_{t<s} h(\alpha, t), \sup_{\beta<\alpha}(h(\beta, s) + 1)\}$ for all α, s. Otherwise, if the requirements in Block α are of P-type, then enumerate x into B; if the requirements in Block α are of Q-type, then enumerate x into A. Let

$$h(\beta, s) = \begin{cases} \max\{\sup_{t<s} h(\beta, t), \sup_{\gamma<\beta}(h(\gamma, s) + 1)\}, & \text{if } \beta \leq \alpha; \\ \max\{\sup_{t<s} h(\beta, t), \sup_{\gamma<\beta}(h(\gamma, s) + 1)\} + s, & \text{if } \beta \geq \alpha + 1. \end{cases}$$

That is, we keep blocks up to Block α, enlarge the next block by s and move the remaining markers accordingly.

6.2 Verification

By Π_1-Foundation, $h(\alpha, s)$ is defined for every s and α. And by the definition of h, for every fixed α, $h(\alpha, s)$ is nondecreasing with respect to s; for every fixed s, $h(\alpha, s)$ is strictly increasing with respect to α.

[2] In the context from here onwards, $\sup F$ is always the least ordinal greater or equal to every element in F, for any M-finite set F of ordinals.

In the construction, we have seen that if
($*$) There is an x enumerated into X at exactly stage s, and there is an $\alpha \leq s$ such that $\beta \geq \alpha + 1$ and $x < \mathbf{r}(\alpha, s)$,
then $h(\beta, s) > \sup_{t<s} h(\beta, t)$. The following lemma states that the converse is also true.

Lemma 6.2 *If $h(\beta, s) > \sup_{t<s} h(\beta, t)$, then ($*$) holds.*

Proof Suppose ($*$) fails. For the sake of contradiction, assume that β is the least such that $h(\beta, s) > \sup_{t<s} h(\beta, t)$. Then $\sup_{\gamma<\beta}(h(\gamma, s) + 1) > \sup_{t<s} h(\beta, t)$. Thus, for some $\gamma_0 < \beta$, $h(\gamma_0, s) \geq \sup_{t<s} h(\beta, t)$. But $h(\gamma_0, s) = \sup_{t<s} h(\gamma_0, t)$. Therefore, $\sup_{t<s} h(\gamma_0, t) \geq \sup_{t<s} h(\beta, t)$. Since for all $t < s$, $h(\gamma_0, t) < h(\gamma, t)$, we have (1) s is limit; (2) $\sup_{t<s} h(\gamma_0, t) = \sup_{t<s} h(\beta, t)$; and (3) $\beta = \gamma_0 + 1$.

Then $h(\beta, s) = \max\{\sup_{t<s} h(\beta, t), h(\gamma_0, s)\} = \max\{\sup_{t<s} h(\beta, t), \sup_{t<s} h(\gamma_0, t)\} = \sup_{t<s} h(\beta, t)$. That is a contradiction. □

Now we define $I = \{\alpha : \exists t \forall s > t(h(\alpha, s) = h(\alpha, t))\}$. We claim that I is downward closed. To see that, we suppose there are ordinals $\beta < \gamma$ such that $\gamma \in I$ but $\beta \notin I$. Let t_0 be a stage such that $\forall s > t_0(h(\gamma, s) = h(\gamma, t_0))$. Since $\beta \notin I$, there is a stage $s > t_0$ such that $h(\beta, s) > \sup_{t<s} h(\beta, t)$. Then by Lemma 6.2, ($*$) holds. Because $\gamma > \beta$, ($*$) also holds if we substitute β by γ in ($*$). Then $h(\gamma, s) > \sup_{t<s} h(\gamma, t)$, deriving a contradiction. Therefore, I might be Ord^M, an ordinal in M, or a Σ_2 cut.

In the argument below, we will show that $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is cofinal in Ord^M. Then we have two conclusions. Firstly, each requirement is assigned with some block from some stage onwards. Secondly, we want to say that $\{\mathbf{r}(\alpha, s) : s \in \mathsf{Ord}^M\}$ is bounded in Ord^M for any $\alpha \in I$. By Lemma 6.2, if $h(\beta, s)$ is stabilized from some stage s_0 onwards, then for each α such that $\alpha < \beta$ and each stage $s > s_0$, there is no $x < \mathbf{r}(a, s)$ enumerated exactly at stage s. Thus,

$$X \cap \mathbf{r}(\alpha, s) = X_s \cap \mathbf{r}(\alpha, s).$$

Fix an $\alpha < \beta$. If $\{\mathbf{r}(\alpha, s) : s \in \mathsf{Ord}^M\}$ is cofinal in Ord^M, then for each $s \in \mathsf{Ord}^M$, elements in $X \cap \mathbf{r}(\alpha, s)$ are determined recursively, and so are those in X, deriving a contradiction (cf. Lemma 6.6).

Lemma 6.3 *$\{\lim_s h(\alpha, s) : \alpha \in I\}$ is regular.*

Proof Fix $\delta \in \mathsf{Ord}^M$. By Lemma 6.1, $\{\alpha : \forall s\,(h(\alpha, s) \leq \delta)\} \subseteq I$. Consider its complement. By Σ_1-Foundation, there is a least ordinal, say α_0, such that $\exists s\,(h(\alpha_0, s) > \delta)$. Note that for any $s > \delta$ and $\alpha < \alpha_0$, $h(\alpha, s) = h(\alpha, \delta)$. Thus, $\{\lim_s h(\alpha, s) : \alpha \in I\} \restriction \delta = \{h(\alpha, \delta) : \alpha < \alpha_0\}$ is M-finite. □

Now suppose $H = \{\lim_s h(\alpha, s) : \alpha \in I\}$ is bounded, and α_0 is the ordinal defined in the proof of Lemma 6.3. Then $\alpha_0 = I$.

Lemma 6.4 *Assume that $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is bounded. Then the ordinal α_0 defined above is not limit.*

Proof Assume that α_0 is limit. Let t be a stage such that for all $\alpha < \alpha_0$, $h(\alpha, t) = \lim_s h(\alpha, s)$. Let s be the least stage such that $h(\alpha_0, s) > h(\alpha_0, t)$. By Lemma 6.2, there is α with $\alpha + 1 < \alpha_0$ such that some $x < \mathbf{r}(\alpha, s)$ is enumerated into X at exactly stage s. Thus, $h(\alpha + 1, s) > h(\alpha + 1, t)$. This is a contradiction. □

By Lemma 6.4, $\alpha_0 = \beta_0 + 1$ for some β_0. Without loss of generality, we assume that β_0 is even. Then $[0, h(\beta_0, s))$ is the limit of Block β_0 and we denote it by B. Let s_0 be the least stage such that $\lim_s h(\beta_0, s) = h(\beta_0, s_0)$.

Lemma 6.5 *Assume that* $\{\lim_s h(\alpha, s) : \alpha \in I\}$ *is bounded. Then* X *is recursive.*

Proof By the construction, for every stage $s > s_0$, $A_{<s} \upharpoonright \mathbf{r}(\beta_0, s) = A \upharpoonright \mathbf{r}(\beta_0, s)$ and from stage $s_0 + 1$ on, both $\mathbf{r}(\beta_0, s)$ and $\mathbf{m}(\beta_0, s)$ are nondecreasing.

Since $\lim_s h(\beta_0 + 1, s) = \infty$, there are cofinally many stages such that $X_s \upharpoonright \mathbf{r}(\beta_0, s) \neq X \upharpoonright \mathbf{r}(\beta_0, s)$. Since X is regular, $\lim_s \mathbf{r}(\beta_0, s) = \infty$. Thus, $\lim_s \mathbf{m}(\beta_0, s) = \infty$. This implies that for every stage $s > s_0$, $e \in B$, $\Phi_e^{A_{<s} \upharpoonright \mathbf{r}(\beta_0, s)}[s] \upharpoonright m(e, s) = X \upharpoonright m(e, s)$.

For every δ, let $s > s_0$ be a stage such that $\mathbf{m}(\beta_0, s) > \delta$. Then $X \upharpoonright \delta = \Phi_e^{A_{<s} \upharpoonright \mathbf{r}(\beta_0, s)}[s] \upharpoonright \delta$, where $e \in B$ is such that $m(e, s) > \delta$. Therefore, $X \upharpoonright \delta = X[s] \upharpoonright \delta$. □

By Lemma 6.5, $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is unbounded in Ord^M. For every $\alpha \in I$, let $B_\alpha = [0, \lim_s h(\alpha, s))$, the limit of Block α.

Lemma 6.6 $X \not\leq_T A$ *and* $X \not\leq_T B$.

Proof We only prove that $X \not\leq_T A$. The proof of $X \not\leq_T B$ is symmetric. For the sake of contradiction suppose $X = \Phi_e^A$, $\alpha \in I$ is even and s_0 is a stage such that $e < \lim_s h(\alpha, s) = h(\alpha, s_0) < \lim_s h(\alpha + 1, s) = h(\alpha + 1, s_0) < s_0$.

By the construction, from stage s_0 on, if $\Phi_e^A[s]$ computes anything, its computation is preserved. Thus, $\Phi_e^A[s] = X \upharpoonright \mathsf{dom}(\Phi_e^A[s])$. Thus, M-finite subsets of both X and $\overline{X}$ can be effectively enumerated via $\Phi_e^A[s]$, $s > s_0$. That implies X is recursive, which is a contradiction. □

Question 6.7 *Is the Sacks Density Theorem true in all models of Level 1-*KPL*?*

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