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Hamiltonian Systems and Sturm–Liouville Equations: Darboux Transformation and Applications

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Abstract. We introduce GBDT version of Darboux transformation for Hamiltonian and Shin–Zettl systems as well as for Sturm–Liouville equations (including indefinite Sturm–Liouville equations). These are the first results on Darboux transformation for general-type Hamiltonian and for Shin–Zettl systems. The obtained results are applied to the corresponding transformations of the Weyl–Titchmarsh functions and to the construction of explicit solutions of dynamical systems, of two-way diffusion equations and of indefinite Sturm–Liouville equations. The energy of the explicit solutions of dynamical systems is expressed (in a quite simple form) in terms of the parameter matrices of GBDT. The insertion of non-real eigenvalues into the spectrum of indefinite Sturm–Liouville operators is studied.

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1. Introduction

This paper is dedicated to the study of the important subclasses of the first order differential systems with a spectral parameter λ. Namely, we consider Hamiltonian systems

$$\frac{d}{dx}y(x,\lambda) = F(x,\lambda)y(x,\lambda), \quad F(x,\lambda) = J\big(\lambda H_1(x) + H_0(x)\big), \tag{1.1}$$

where

$$J^* = -J, \quad H_1(x)^* = H_1(x), \quad H_0(x) = H_0(x)^*, \quad H_1(x) \geq 0; \tag{1.2}$$

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and so called Shin–Zettl systems

$$\frac{d}{dx}y(x,\lambda) = F(x,\lambda)y(x,\lambda), \quad F(x,\lambda) = \begin{bmatrix} r_1(x) & p(x)^{-1} \\ q(x) - \lambda\omega(x) & r_2(x) \end{bmatrix}. \quad (1.3)$$

Here J^* is the conjugate transpose of the matrix J. We assume that the $m \times m$ ($m \in \mathbb{N}$) matrix functions $H_1(x)$ and $H_0(x)$ in (1.1) and the functions p^{-1}, q, r_1, r_2 and ω in (1.3) are locally summable on $[0, \ell)$ ($\ell \leq \infty$). The matrix function F in (1.3) is the 2×2 Shin–Zettl matrix of general form (see, e.g., § 2 in [14] or in [15]). We note that Shin–Zettl differential expressions were introduced in [50,53] and were actively studied in regularization and spectral theories (see the books [1,54], papers [14,15], recent surveys [37,55] and various references therein). The Lagrange-symmetric case

$$\omega = \overline{\omega}, \quad p = \overline{p}, \quad q = \overline{q}, \quad r_1 = -\overline{r_2} \quad (1.4)$$

and the Lagrange-J-symmetric case

$$r_1 = -r_2 \quad (1.5)$$

are of special interest [15]. Here $\overline{\mu}$ stands for the value which is complex conjugate to μ.

The entries of the 2×1 vector function y in (1.3) are denoted by y_1 and y_2. When $r_1 \equiv r_2 \equiv 0$, we rewrite (1.3) in the form

$$y_1' = p^{-1}y_2, \quad y_2' = (q - \lambda\omega)y_1 \quad \left(y_k' = \frac{d}{dx}y_k\right), \quad (1.6)$$

which is equivalent to the *Sturm–Liouville equation*

$$-\big(p(x)u'(x,\lambda)\big)' + q(x)u(x,\lambda) = \lambda\omega(x)u(x,\lambda), \quad (1.7)$$

where $u = y_1$. If $\omega = \overline{\omega}$, $p = \overline{p}$ and ω or p change signs, one speaks about *indefinite Sturm–Liouville problem.* Quasi-derivatives related to the quasi-derivatives generated by Shin–Zettl systems are used in the study of important modifications of Schrödinger-type operators (see, e.g., [13,49] and references therein) including Schrödinger-type operators with distributional potentials [13].

On the other hand, Lagrange-symmetric Shin–Zettl systems, where $\omega \geq 0$, form also a subclass of Hamiltonian systems. See, for instance, [22] on the representation (1.1), (1.2) of Hamiltonian systems and the equivalence of the definite Sturm–Liouville equation to a certain subclass of Hamiltonian systems. We note that the book [2] by Atkinson, the papers by Hinton and Shaw as well the Kac–Krein supplement [24] (to the translation of [2]) presented seminal developments in the theory of Hamiltonian systems and Sturm–Liouville equations. (For recent references on Hamiltonian systems see, e.g., [25,38,46,51].) In some works, conditions (3.1) are added in the definition of Hamiltonian systems but these conditions are absent in [22] and they are not essential for Darboux transformations, which we will construct here, as well.

In this paper we construct our GBDT version of the Bäcklund–Darboux transformation (see the results and references in [42,44,46]) for the cases of Hamiltonian and Shin–Zettl systems in order to study perturbations of these systems and corresponding transformations of the Weyl–Titchmarsh (or simply Weyl) functions. We construct explicit solutions of the perturbed systems as well. Several versions of Bäcklund–Darboux transformations (see, e.g., [9, 21, 35, 46] and references therein) are a well-known tool for the construction of explicit solutions of linear and integrable nonlinear equations. GBDT as well as Crum–Krein and commutation methods (which are related to Bäcklund–Darboux transformations) are also essential in the study of Weyl–Titchmarsh theory and important spectral problems [10, 12, 17, 18, 20, 29, 32, 36, 45].

As far as we know, neither Bäcklund–Darboux transformations nor commutation methods were applied to general-type Hamiltonian systems (1.1) and to Shin–Zettl systems (1.3) before (although commutation and Bäcklund–Darboux transformations for such important particular cases as Schrödinger equations, canonical systems and related Dirac equations are well-known). We mention an interesting paper [5] on Kummer–Liouville transformation for Shin–Zettl systems but that transformation is different and was applied with different purposes.

Darboux transformation for general-type Hamiltonian systems is introduced in Sect. 2. The corresponding transformations of the Weyl functions are considered in Sect. 3. GBDT for Shin–Zettl systems and Sturm–Liouville equations is introduced in Sects. 4–6. Explicit solutions of dynamical systems and of two-way diffusion equations are constructed in Sect. 7. The insertion of real and non-real eigenvalues is discussed in Sect. 8. Finally, explicit solutions of indefinite Sturm–Liouville equations (and explicit expressions for generalised eigenfunctions) are considered in Sect. 9.

As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex plane, $\mathbb{C}_+$ is the open upper half-plane $\{\lambda\colon \Im(\lambda) > 0\}$ and $\mathbb{C}_-$ is the open lower half-plane $\{\lambda\colon \Im(\lambda) < 0\}$. The notation I_n stands for the $n\times n$ identity matrix, H^* is the conjugate transpose of the matrix H, the inequality $H \geq 0$ means that $H = H^*$ and that all the eigenvalues of the matrix H are nonnegative.

2. GBDT for Hamiltonian Systems

1. Our GBDT version of Bäcklund–Darboux transformation for system (1.1) is a particular case of GBDT for systems with rational dependence on spectral parameter (see, e.g., [44] or [46, Sect. 7.2]). We start with introducing GBDT for the first order system of m differential equations with a linear dependence on the spectral parameter ($m \in \mathbb{N}$):

$$y'(x,\lambda) = F(x,\lambda)y(x,\lambda), \quad F(x,\lambda) = -\big(\lambda Q_1(x) + Q_0(x)\big). \tag{2.1}$$

For that purpose we fix some initial system (2.1) (i.e., some $m \times m$ matrix functions $Q_1(x)$ and $Q_0(x)$, which are locally summable on $[0,\ell)$), an integer $n \in \mathbb{N}$ and five parameter matrices, namely, $n \times n$ matrices A_1, A_2 and $S(0)$, and $n \times m$ matrices $\Pi_1(0)$ and $\Pi_2(0)$ such that the matrix identity

$$A_1 S(0) - S(0) A_2 = \Pi_1(0)\Pi_2(0)^* \tag{2.2}$$

holds. Matrix functions $\Pi_1(x)$, $\Pi_2(x)$ and $S(x)$ are introduced by their initial values $\Pi_1(0)$, $\Pi_2(0)$, $S(0)$ and differential equations

$$\Pi_1' = A_1\Pi_1 Q_1 + \Pi_1 Q_0, \quad (\Pi_2^*)' = -Q_1\Pi_2^* A_2 - Q_0\Pi_2^*, \quad S' = \Pi_1 Q_1 \Pi_2^*. \tag{2.3}$$

The identity

$$A_1 S(x) - S(x) A_2 = \Pi_1(x)\Pi_2(x)^*, \tag{2.4}$$

for all $x \in [0, \ell)$, is a particular case of [46, f-la (7.18)] and easily follows from (2.2) and (2.3).

When we deal with $S(x)^{-1}$, our further statements are valid in the points of invertibility of $S(x)$. The questions of invertibility of $S(x)$ are discussed in our sections separately (see, e.g., Remarks 2.3, 8.2, 9.4 and formulas (9.7) and (9.8)).

According to the subcase $r = 1,\ l = 0$ of [46, Theor. 7.4], the so called Darboux matrix for system (2.1) is given by the formula

$$w_A(x, \lambda) = I_m - \Pi_2(x)^* S(x)^{-1}(A_1 - \lambda I_n)^{-1}\Pi_1(x). \tag{2.5}$$

More precisely, [46, Theor. 7.4] yields that w_A satisfies the following equation

$$\frac{d}{dx} w_A(x, \lambda) = \widetilde{F}(x, \lambda) w_A(x, \lambda) - w_A(x, \lambda) F(x, \lambda), \tag{2.6}$$

where

$$\widetilde{F}(x, \lambda) := -\big(\lambda Q_1(x) + \widetilde{Q}_0(x)\big), \tag{2.7}$$

$$\widetilde{Q}_0(x) := Q_0(x) - \big(Q_1(x)X(x) - X(x)Q_1(x)\big), \tag{2.8}$$

$$X(x) := \Pi_2(x)^* S(x)^{-1}\Pi_1(x). \tag{2.9}$$

We note that (in view of (2.4)) the matrix function $w_A(\lambda)$ of the form (2.5) is (for each x) the so called transfer matrix function in Lev Sakhnovich form (see [46–48] and references therein).

System $y' = \widetilde{F} y$ is called the transformed (GBDT-transformed) system (recall that (2.1) is the initial system). An important step in the proof of (2.6) is the proof of the equation

$$\big(\Pi_2^* S^{-1}\big)' = -Q_1\Pi_2^* S^{-1} A_1 - \widetilde{Q}_0 \Pi_2^* S^{-1}. \tag{2.10}$$

See [46, f-la (7.61)] for the general formula, of which (2.10) is a particular case. We will use (2.10) in Sect. 7.1.

Formula (2.6) implies the following theorem.

Theorem 2.1. *Let $y(x, \lambda)$ satisfy system (2.1) and let w_A be given by (2.5), where the matrix functions Π_1, Π_2 and S are determined by (2.3) and identity (2.2) holds. Then the function*

$$\widetilde{y}(x, \lambda) := w_A(x, \lambda) y(x, \lambda) \tag{2.11}$$

satisfies, in the points of invertibility of $S(x)$, another (transformed) first order system

$$\frac{d}{dx}\widetilde{y}(x,\lambda) = \widetilde{F}(x,\lambda)\widetilde{y}(x,\lambda), \tag{2.12}$$

where $\widetilde{F}(x,\lambda)$ *is given by* (2.7)–(2.9).

2. The most important subcase of the considered above transformations (GBDT-transformations) is the subcase of the initial system (2.1) such that

$$Q_1(x) = -JH_1(x), \quad Q_0(x) = -JH_0(x), \tag{2.13}$$

$$J^* = -J, \quad H_1(x)^* = H_1(x), \quad H_0(x) = H_0(x)^*. \tag{2.14}$$

In that subcase we deal with the system (1.1), where all the conditions (1.2) on Hamiltonian system, excluding the nonnegativity condition $H_1(x) \geq 0$, hold. Further in the paragraphs 2 and 3 we assume that the equalities (2.13) and (2.14) are valid. Since we do not require in paragraph 2 that $H_1 \geq 0$, we may apply the corresponding results not only to Hamiltonian systems but also to all Lagrange-symmetric Shin–Zettl systems and to indefinite Sturm–Liouville equations.

We omit indices in A_1 and Π_1 and set

$$A = A_1, \quad \Pi = \Pi_1; \quad A_2 = A^*, \quad S(0) = S(0)^*, \quad \Pi_2(0) = -\Pi(0)J. \tag{2.15}$$

Using (2.13)–(2.15) we rewrite the first and second equations in (2.3), correspondingly, in the forms

$$(-\Pi J)' = -A(-\Pi J)H_1 J - (-\Pi J)H_0 J, \quad (\Pi_2)' = -A\Pi_2 H_1 J - \Pi_2 H_0 J.$$

Thus, the equations on $-\Pi J$ and on Π_2 coincide, and, in view of the equality $\Pi_2(0) = -\Pi(0)J$ we obtain $\Pi_2(x) \equiv -\Pi(x)J$. In this way, Eq. (2.3) is reduced to the equation

$$\Pi'(x) = -A\Pi(x)JH_1(x) - \Pi(x)JH_0(x), \quad S'(x) = \Pi(x)JH_1(x)J^*\Pi(x)^*. \tag{2.16}$$

Since we assume in (2.15) that $S(0) = S(0)^*$, the second equation in (2.16) yields $S(x) = S(x)^*$. Thus, we have

$$\Pi_2(x) \equiv -\Pi(x)J, \quad S(x) = S(x)^*. \tag{2.17}$$

Now, the matrix identity (2.4) and Darboux matrix (2.5) are rewritten in the form

$$AS(x) - S(x)A^* = \Pi(x)J\Pi(x)^*, \tag{2.18}$$

$$w_A(x,\lambda) = I_m - J\Pi(x)^*S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x). \tag{2.19}$$

Moreover, using (2.13) and the equalities $\Pi_2(x) \equiv -\Pi(x)J$ and $J^* = -J$, we rewrite (2.7)–(2.9) in the form

$$\widetilde{F}(x,\lambda) = J\big(\lambda H_1(x) + \widetilde{H}_0(x)\big), \quad \widetilde{H}_0(x) = H_0(x) + Z(x), \tag{2.20}$$

$$Z(x) := \Pi(x)^*S(x)^{-1}\Pi(x)JH_1(x) + H_1(x)J^*\Pi(x)^*S(x)^{-1}\Pi(x). \tag{2.21}$$

Formulas (2.14), (2.20) and (2.21) imply that $\widetilde{H}_0 = \widetilde{H}_0^*$, that is, $\widetilde{F}$ has the same form as F. Hence, the next proposition follows from Theorem 2.1.

Proposition 2.2. *Let $y(x,\lambda)$ satisfy system (1.1) (such that (2.14) holds), and let a triple $\{A, S(0) = S(0)^*, \Pi(0)\}$ of parameter matrices satisfying (2.18) at $x = 0$ be given. Introduce $w_A(x,\lambda)$ by (2.19), where the matrix functions $\Pi(x)$ and $S(x)$ are determined by (2.16).*

Then the function $\widetilde{y}(x,\lambda) = w_A(x,\lambda)y(x,\lambda)$ satisfies, in the points of invertibility of $S(x)$, another (transformed) system of the same form as (1.1), namely,

$$\frac{d}{dx}\widetilde{y}(x,\lambda) = \widetilde{F}(x,\lambda)\widetilde{y}(x,\lambda), \tag{2.22}$$

where $\widetilde{F}(x,\lambda)$ is given by (2.20), (2.21) and the equality $\widetilde{H}_0 = \widetilde{H}_0^*$ holds.

If the inequalities $H_1(x) \geq 0$ and $S(0) > 0$ are valid, the systems (1.1) and (2.22) are Hamiltonian.

Remark 2.3. If system (1.1), (2.14) is Hamiltonian (i.e., $H_1(x) \geq 0$) and, in addition, the inequality $S(0) > 0$ holds, formula (2.16) shows that $S(x) > 0$ for all $x \in [0, \ell)$. Therefore, $S(x)$ is invertible on $[0, \ell)$. In particular, it follows that the system (2.22) is, indeed, Hamiltonian.

3. If in the system (1.1) we have $J = -J^* = -J^{-1}$ and $H_0 \equiv 0$, we come to the important class of canonical systems. See GBDT for canonical system and its applications to Weyl–Titchmarsh theory in [43].

For the case of Hamiltonian systems with invertible J we can (similar to the case of canonical systems) consider transformation slightly different from (2.20), (2.21). More precisely, we introduce matrix functions $\widehat{w}(x)$ and $v(x,\lambda)$ by the formulas

$$\widehat{w}'(x) = -\widehat{w}(x)JZ(x), \quad \widehat{w}(0) = I_m; \quad v(x,\lambda) = \widehat{w}(x)w_A(x,\lambda) \tag{2.23}$$

with Z of the form (2.21). It is easy to see that $\widehat{w}(x)J\widehat{w}(x)^* = J$, and so

$$\widehat{w}(x)^{-1} = J\widehat{w}(x)^*J^{-1} = J^*\widehat{w}(x)^*(J^*)^{-1}. \tag{2.24}$$

In view of Proposition 2.2 and relations (2.23) and (2.24), if $y(x,\lambda)$ satisfies (1.1), then the matrix function $\widehat{y}(x,\lambda) = v(x,\lambda)y(x,\lambda)$ satisfies the system

$$\frac{d}{dx}\widehat{y}(x,\lambda) = \widehat{F}(x,\lambda)\widehat{y}(x,\lambda), \quad \widehat{F}(x,\lambda) = J\big(\lambda\widehat{H}_1(x) + \widehat{H}_0(x)\big), \tag{2.25}$$

where

$$\widehat{H}_1 = J^{-1}\widehat{w}JH_1\widehat{w}^{-1} = J^{-1}\widehat{w}JH_1J^*\widehat{w}^*(J^*)^{-1} = \widehat{H}_1^*, \tag{2.26}$$

$$\widehat{H}_0 = J^{-1}\widehat{w}J(\widetilde{H}_0 - Z)\widehat{w}^{-1} = J^{-1}\widehat{w}JH_0J^*\widehat{w}^*(J^*)^{-1} = \widehat{H}_0^*. \tag{2.27}$$

In the special case $H_0 = \mathrm{i}cJ^{-1}$ ($c = \overline{c}$), the formula (2.27) is simplified and we obtain $\widehat{H}_0 \equiv \mathrm{i}cJ^{-1}$.

3. Darboux Transformations of Weyl–Titchmarsh Functions

In his important paper [30], Krall introduced Weyl–Titchmarsh (or simply Weyl) $M(\lambda)$-functions of Hamiltonian systems in the classical terms of "Weyl circle" inequalities. Here, Weyl circles of system (1.1) on the intervals $[0, \ell']$ ($\ell' < \ell$) and the values λ in the upper half-plane $\lambda \in \mathbb{C}_+$ (i.e., $\Im(\lambda) > 0$) are considered. The Weyl circles in the lower half-plane $\mathbb{C}_-$ are treated in a quite similar way and we omit that case.

Krall required that m is even and that J in (1.1) has a special form:

$$m = 2r \quad (r \in \mathbb{N}), \quad J = \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}. \tag{3.1}$$

In fact, Hamiltonian system in [30] is written in a slightly different from (1.1) way and our J^* stands for J in an equivalent to (1.1) system in [30]. Rewriting correspondingly the inequality for the Weyl circle (of matrices $M(\lambda)$ with $\lambda \in \mathbb{C}_+$) from [30, p. 670], we obtain

$$\mathrm{i} \begin{bmatrix} I_r & M(\lambda)^* \end{bmatrix} Y(\ell', \lambda)^* J Y(\ell', \lambda) \begin{bmatrix} I_r \\ M(\lambda) \end{bmatrix} \leq 0. \tag{3.2}$$

Here $Y(x, \lambda)$ is the fundamental $m \times m$ solution of the Hamiltonian system (1.1) (such that (1.2) and (3.1) are valid), normalized by the initial condition

$$Y(0, \lambda) = E \quad (EJ = JE, \quad E^*E = I_m). \tag{3.3}$$

According to Proposition 2.2, the fundamental solution $\widetilde{Y}(x, \lambda)$ (normalized by $\widetilde{Y}(0, \lambda) = E$) of the transformed Hamiltonian system (2.22) is given by the formula

$$\widetilde{Y}(x, \lambda) = w_A(x, \lambda) Y(x, \lambda) E^* w_A(0, \lambda)^{-1} E. \tag{3.4}$$

Let us set

$$\mathcal{U}(\lambda) = \{\mathcal{U}_{ij}(\lambda)\}_{i,j=1}^2 := E^* w_A(0, \lambda) E, \tag{3.5}$$

where $\mathcal{U}_{ij}(\lambda)$ are $r \times r$ blocks of $\mathcal{U}$. In view of (3.4) and (3.5), the Weyl circle (of matrices $\widetilde{M}(\lambda)$) for the transformed system on $[0, \ell']$ and for $\lambda \in \mathbb{C}_+$ is determined by the inequality

$$\mathrm{i} \begin{bmatrix} I_r & \widetilde{M}(\lambda)^* \end{bmatrix} \left(\mathcal{U}(\lambda)^{-1}\right)^* Y(\ell', \lambda)^* w_A(x, \lambda)^* J w_A(x, \lambda) Y(\ell', \lambda) \mathcal{U}(\lambda)^{-1} \begin{bmatrix} I_r \\ \widetilde{M}(\lambda) \end{bmatrix} \leq 0. \tag{3.6}$$

Relations (2.18) and (2.19) yield the following identity [46, f-la (1.88)]:

$$\begin{aligned} &\mathrm{i} w_A(x, \lambda)^* J w_A(x, \lambda) \\ &\quad = \mathrm{i}J + \mathrm{i}(\lambda - \overline{\lambda}) \Pi(x)^* (A^* - \overline{\lambda} I_n)^{-1} S(x)^{-1} (A - \lambda I_n)^{-1} \Pi(x). \end{aligned} \tag{3.7}$$

In this section we consider Hamiltonian systems and assume that $S(0) > 0$. Hence, according to Remark 2.3 we have $S(x) > 0$. Now, it is immediate from (3.7) that

$$\mathrm{i} w_A(x, \lambda)^* J w_A(x, \lambda) \leq \mathrm{i}J \quad (\lambda \in \mathbb{C}_+). \tag{3.8}$$

Formula (3.8) implies that

$$\mathrm{i}\left[I_r\ \widetilde{M}(\lambda)^*\right]\big(\mathcal{U}(\lambda)^{-1}\big)^*Y(\ell',\lambda)^*w_A(x,\lambda)^*Jw_A(x,\lambda)Y(\ell',\lambda)\mathcal{U}(\lambda)^{-1}\begin{bmatrix}I_r\\ \widetilde{M}(\lambda)\end{bmatrix}$$
$$\leq \mathrm{i}\left[I_r\ \widetilde{M}(\lambda)^*\right]\big(\mathcal{U}(\lambda)^{-1}\big)^*Y(\ell',\lambda)^*JY(\ell',\lambda)\mathcal{U}(\lambda)^{-1}\begin{bmatrix}I_r\\ \widetilde{M}(\lambda)\end{bmatrix} \qquad (3.9)$$

Using (3.9), we derive the next theorem.

Theorem 3.1. *Let Hamiltonian system* (1.1) (*such that* (1.2) *and* (3.1) *are valid*) *be given. Let its GBDT transformation be determined by the triple of matrices* $\{A, S(0), \Pi(0)\}$ *such that* $S(0) > 0$ *and that the matrix identity*

$$AS(0) - S(0)A^* = \Pi(0)J\Pi(0)^* \qquad (3.10)$$

holds. Assume that $M(\lambda)$ $(\lambda \in \mathbb{C}_+)$ *belongs to the Weyl circle* (3.2) *of the system* (1.1) *and that*

$$\det\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)M(\lambda)\big) \neq 0, \qquad (3.11)$$

where $\mathcal{U}$ *is defined in* (3.5)*. Then*

$$\widetilde{M}(\lambda) = \big(\mathcal{U}_{21}(\lambda) + \mathcal{U}_{22}(\lambda)M(\lambda)\big)\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)M(\lambda)\big)^{-1} \qquad (3.12)$$

belongs to the Weyl circle of the transformed system.

Proof. Taking into account (3.11) and (3.12), we obtain

$$\begin{bmatrix}I_r\\ \widetilde{M}(\lambda)\end{bmatrix} = \mathcal{U}(\lambda)\begin{bmatrix}I_r\\ M(\lambda)\end{bmatrix}\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)M(\lambda)\big)^{-1}. \qquad (3.13)$$

Now, substitute (3.13) into the right-hand side of (3.9) and use (3.2) in order to see that (3.6) is valid. □

According to [30, p. 671], we have $\mathrm{i}\big(M(\lambda) - M(\lambda)^*\big) \leq 0$. Moreover, we have

$$\mathrm{i}\big(M(\lambda) - M(\lambda)^*\big) < 0, \qquad (3.14)$$

if only $\int_0^{\ell'} y(x,\lambda)^*H_1(x)y(x,\lambda)dx > 0$ for each nontrivial solution y of (1.1).

Remark 3.2. If (3.14) is valid, then the inequality (3.11) holds automatically. Indeed, if $\det\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)M(\lambda)\big) = 0$, then there is a vector $f \neq 0$ such that $\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)M(\lambda)\big)f = 0$. Therefore, recalling that J has the form (3.1), we obtain

$$\mathrm{i}f^*\left[I_r\ M(\lambda)^*\right]\mathcal{U}(\lambda)^*J\mathcal{U}(\lambda)\begin{bmatrix}I_r\\ M(\lambda)\end{bmatrix}f = 0 \quad (f \neq 0). \qquad (3.15)$$

On the other hand, relations (3.5) and (3.8) (together with the properties of E from (3.3)) imply that $\mathcal{U}(\lambda)^*J\mathcal{U}(\lambda) \leq \mathrm{i}J$. Hence, using (3.14), we derive

$$\mathrm{i}\left[I_r\ M(\lambda)^*\right]\mathcal{U}(\lambda)^*J\mathcal{U}(\lambda)\begin{bmatrix}I_r\\ M(\lambda)\end{bmatrix} \leq \mathrm{i}\left[I_r\ M(\lambda)^*\right]J\begin{bmatrix}I_r\\ M(\lambda)\end{bmatrix} < 0, \qquad (3.16)$$

which contradicts (3.15).

In the limit point case (see, e.g., the discussions in [23,31]) there is a unique holomorphic in $\mathbb{C}_+$ Weyl function $\mathcal{M}(\lambda)$ the values of which belong to all the Weyl circles (3.2) such that $\ell' < \ell$ $(\lambda \in \mathbb{C}_+)$. We note that $\mathcal{M}(\lambda)$ is the limit of the values of $M(\lambda)$ when ℓ' tends to ℓ. Thus, formula (3.12) shows that

$$\widetilde{\mathcal{M}}(\lambda) := \big(\mathcal{U}_{21}(\lambda) + \mathcal{U}_{22}(\lambda)\mathcal{M}(\lambda)\big)\big(\mathcal{U}_{11}(\lambda) + \mathcal{U}_{12}(\lambda)\mathcal{M}(\lambda)\big)^{-1} \quad (3.17)$$

is a Weyl function of the transformed system considered on $[0, \ell)$.

4. GBDT for Shin–Zettl Systems

Shin–Zettl systems (1.3) present (as well as Hamiltonian systems) an important subclass of systems (2.1). Matrices Q_1 and Q_2, in the case of Shin–Zettl systems, have the form

$$Q_1(x) = \begin{bmatrix} 0 & 0 \\ \omega(x) & 0 \end{bmatrix}, \quad Q_0(x) = -\begin{bmatrix} r_1(x) & p(x)^{-1} \\ q(x) & r_2(x) \end{bmatrix}. \quad (4.1)$$

Recall that GBDT is determined by the parameter matrices A_1, A_2, $S(0)$, $\Pi_1(0)$ and $\Pi_2(0)$ such that (2.2) holds. For the Shin–Zettl systems, we have $m = 2$, and so matrices $\Pi_1(0)$ and $\Pi_2(0)$ are $n \times 2$ matrices. Using the second equality in (1.3) and the first equality in (4.1), we rewrite $\widetilde{F}$ given by (2.7)–(2.9) in the Shin–Zettl form

$$\widetilde{F}(x,\lambda) = \begin{bmatrix} \widetilde{r}_1(x) & \widetilde{p}(x)^{-1} \\ \widetilde{q}(x) - \lambda\widetilde{\omega}(x) & \widetilde{r}_2(x) \end{bmatrix}, \quad \widetilde{\omega} = \omega, \quad \widetilde{p} = p; \quad (4.2)$$

$$\widetilde{r}_1 = r_1 - \omega X_{12}, \quad \widetilde{r}_2 = r_2 + \omega X_{12}, \quad \widetilde{q} = q + \omega(X_{11} - X_{22}). \quad (4.3)$$

where $X_{ik}(x)$ are the entries of $X(x)$. Now, the following proposition is immediate from Theorem 2.1.

Proposition 4.1. *Let $y(x,\lambda)$ satisfy Shin–Zettl system (1.3) and let w_A be given by (2.5), where the matrix functions Π_1, Π_2 and S are determined by (2.3) and identity (2.2) holds. Then the function $\widetilde{y}(x,\lambda) = w_A(x,\lambda)y(x,\lambda)$ satisfies, in the points of invertibility of $S(x)$, the transformed Shin–Zettl system (2.12), where $\widetilde{F}(x,\lambda)$ is given by (4.2) and (4.3).*

The next corollary easily follows from Proposition 4.1.

Corollary 4.2. *Let the conditions of Proposition 4.1 hold and let the initial system (1.3) be Lagrange-J-symmetric (i.e., let (1.5) be valid). Then the transformed system is Lagrange-J-symmetric as well, that is, the equality $\widetilde{r}_1 = -\widetilde{r}_2$ holds.*

In the next section, we consider the Lagrange-symmetric case (i.e., the case (1.4)).

5. Lagrange-Symmetric Case

Further we assume that (1.4) is fulfilled and rewrite (4.1) for that case:

$$Q_1(x) = \begin{bmatrix} 0 & 0 \\ \omega(x) & 0 \end{bmatrix}, \quad Q_0(x) = -\begin{bmatrix} r(x) & p(x)^{-1} \\ q(x) & -\overline{r(x)} \end{bmatrix}, \tag{5.1}$$

$$r(x) := r_1(x) = -\overline{r_2(x)}. \tag{5.2}$$

Now, system (1.3) may be rewritten as the quasi-differential equation:

$$-\left(u^{[1]}\right)' - \overline{r}u^{[1]} + qu = \lambda\omega u, \qquad u^{[1]} := p(u' - ru), \tag{5.3}$$

where $y_1(x) = u(x)$, $y_2(x) = u^{[1]}(x)$ and the quasi-differential expression

$$Mu = -\left(u^{[1]}\right)' - \overline{r}u^{[1]} + qu - \lambda\omega u$$

is symmetric (see, e.g., [15]). See also [7,39,54] and references therein on symmetric expressions $(-\left(u^{[1]}\right)' - \overline{r}u^{[1]} + qu)/\omega$ in the weighted spaces $L^2_{|\omega|}[0,\ell)$ and $L^2_{\omega}[0,\ell)$. Using the quasi-derivative $u^{[1]}$ one may consider Sturm–Liouville equations (including self-adjoint Sturm–Liouville equations) with non-smooth coefficients (see, e.g., the discussions in [53, p. 455] and in [54, p. 25]).

We note that Q_1 and Q_0 given by (5.1) admit representation (2.13), where

$$J = \mathrm{i}\sigma_2, \quad H_1(x) = \begin{bmatrix} \omega(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad H_0(x) = \begin{bmatrix} -q(x) & \overline{r(x)} \\ r(x) & p(x)^{-1} \end{bmatrix}, \tag{5.4}$$

$\sigma_2 := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$ is a Pauli matrix, and (2.14) holds. In fact, conditions (2.13) and (2.14) (in the Shin–Zettl case and with $J = \mathrm{i}\sigma_2$) are equivalent to the conditions (1.4) of Lagrange symmetry. (Clearly, when $\omega \geq 0$ we deal with a subclass of Hamiltonian systems.) Thus, omitting the indices in A_1 and Π_1 and rewriting (2.15) in the form

$$A = A_1, \quad \Pi = \Pi_1; \quad A_2 = A^*, \quad S(0) = S(0)^*, \quad \Pi_2(0) = -\mathrm{i}\Pi(0)\sigma_2, \tag{5.5}$$

we see that the formulas of §2 in Sect. 2 are valid for Lagrange-symmetric case.

Since $\Pi_2(x) = -\mathrm{i}\Pi(x)\sigma_2$, formula (2.9) for X may be rewritten as

$$X(x) = J\Pi(x)^* S(x)^{-1}\Pi(x), \quad J = \mathrm{i}\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5.6}$$

and we obtain

$$X_{12}(x) = \overline{X_{12}(x)}, \quad X_{22}(x) = -\overline{X_{11}(x)}. \tag{5.7}$$

Recall that $\Pi(x)$ and $S(x)$ are given by the equations

$$\Pi' = -A\Pi JH_1 - \Pi JH_0, \quad S' = \Pi JH_1J^*\Pi^*, \quad J = \mathrm{i}\sigma_2. \tag{5.8}$$

Formula (2.19) for the Darboux matrix takes the form

$$w_A(x,\lambda) = I_2 - \mathrm{i}\sigma_2\Pi(x)^* S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x). \tag{5.9}$$

Using (5.4) and (5.7), we derive from Propositions 2.2 and 4.1 the next corollary.

Corollary 5.1. *Assume that the initial Shin–Zettl system is Lagrange-symmetric (i.e., that* (1.4) *holds). Let the matrices* A, $\Pi(0)$ *and* $S(0)$ *be chosen so that* $S(0) = S(0)^*$ *and* $AS(0) - S(0)A^* = \mathrm{i}\Pi(0)\sigma_2\Pi(0)^*$, *and let* $\Pi(x)$, $S(x)$ *and* $X(x)$ *be determined by* (5.8) *and* (5.6), *respectively.*

Then the corresponding transformed Shin–Zettl system $\widetilde{y}'(x,\lambda) = \widetilde{F}(x,\lambda)\widetilde{y}(x,\lambda)$ *is given by* (4.2), *where*

$$\widetilde{r}_1 = -\overline{\widetilde{r}_2} = r - \omega X_{12}, \quad \widetilde{q} = q + \omega(X_{11} + \overline{X_{11}}). \tag{5.10}$$

This transformed system is Lagrange-symmetric as well. Moreover, the function $\widetilde{y}(x,\lambda) = w_A(x,\lambda)y(x,\lambda)$, *where* w_A *has the form* (5.9) *and* y *is a solution of the initial Shin–Zettl system, satisfies the transformed system.*

6. Sturm–Liouville Equations

In this section we consider Sturm–Liouville equation (1.7). GBDT for its particular case (namely, for Schrödinger equation where $p \equiv \omega \equiv 1$) was dealt with in [19] but the general equation (1.7) contains other interesting subcases, where GBDT could be useful as well.

Proposition 6.1. *Let the function* $p\omega$ *be differentiable and its derivative* $(p\omega)'$ *as well as the functions* p^{-1}, q *and* ω *be locally summable on* $[0,\,\ell)$. *Assume that*

$$\omega = \overline{\omega}, \quad p = \overline{p}, \quad q = \overline{q}, \quad r \equiv 0, \tag{6.1}$$

and set

$$\widetilde{y}(x,\lambda) = w_A(x,\lambda)y(x,\lambda), \tag{6.2}$$

where w_A *is given by the relations* (5.9) *and* (5.8), H_0 *and* H_1 *(in* (5.8)*) are given by* (5.4) *and* y *satisfies the initial Lagrange-symmetric Shin–Zettl system*

$$y'(x,\lambda) = J\big(\lambda H_1(x) + H_0(x)\big)y(x,\lambda). \tag{6.3}$$

Then the entry $\widetilde{y}_1$ *of* $\widetilde{y}$ *satisfies the transformed Sturm–Liouville equation*

$$-\big(p(x)\widetilde{y}_1'(x,\lambda)\big)' + \breve{q}(x)\widetilde{y}_1(x,\lambda) = \lambda\omega(x)\widetilde{y}_1(x,\lambda), \tag{6.4}$$

where

$$\breve{q} = q + 2\omega(X_{11} - X_{22}) + 2p(\omega X_{12})^2 - (p\omega)'X_{12}, \tag{6.5}$$

and X_{ik} *are the entries of* X *given by* (5.6).

Proof. Recall that in Sect. 5 we rewrote Shin–Zettl system in the form (5.3) where $u = y_1$. In the notations of the transformed system it means

$$-\big(p(\widetilde{y}_1' - \widetilde{r}\widetilde{y}_1)\big)' - \overline{\widetilde{r}}p(\widetilde{y}_1' - \widetilde{r}\widetilde{y}_1) + \widetilde{q}\widetilde{y}_1 = \lambda\omega\widetilde{y}_1, \tag{6.6}$$

where $\widetilde{r} := \widetilde{r}_1(x) = -\overline{\widetilde{r}_2(x)}$. Using the identity $r_1 \equiv r_2 \equiv 0$ and equalities (4.3) and (5.7) we present (6.6) in the form

$$-\big(p\widetilde{y}_1'\big)' - \big(p\omega X_{12}\widetilde{y}_1\big)' + p\omega X_{12}\widetilde{y}_1' + p(\omega X_{12})^2\widetilde{y}_1 \\ +\big(q + \omega(X_{11} - X_{22})\big)\widetilde{y}_1 = \lambda\omega\widetilde{y}_1, \tag{6.7}$$

which is equivalent to (6.4) with

$$\breve{q} = q + \omega(X_{11} - X_{22}) + p(\omega X_{12})^2 - (p\omega)' X_{12} - p\omega X_{12}'. \tag{6.8}$$

Finally, in order to show that the functions $\breve{q}$ given by (6.5) and (6.8) coincide, let us differentiate X_{12}. Taking into account (5.6) and (5.8), we obtain:

$$X' = J(H_1 J\Pi^* A^* S^{-1}\Pi + H_0 J\Pi^* S^{-1}\Pi) - J\Pi^* S^{-1}\Pi J H_1 J^* \Pi^* S^{-1}\Pi \\ - J\Pi^* S^{-1}(A\Pi J H_1 + \Pi J H_0).$$

In particular, for X_{12} we obtain

$$X_{12}' = p^{-1}(X_{22} - X_{11}) - \omega X_{12}^2. \tag{6.9}$$

Here we again took into account that $r_1 \equiv r_2 \equiv 0$. Equalities (6.8) and (6.9) imply (6.5). □

Remark 6.2. In view of (5.7), (6.1) and (6.5), the equality $\Im(\breve{q}) \equiv 0$ is valid. Thus, the coefficients of the transformed Sturm–Liouville equation (6.4) are real-valued. It is easy to see that the function $\breve{q}$ is locally summable on $[0, \ell)$ if the conditions of Proposition 6.1 hold and $S(x)$ is invertible on $[0, \ell)$.

7. Dynamical Systems

7.1. Dynamical Systems

Formally applying Laplace transform to the system (1.1) (satisfying (2.14)), we come to the interesting dynamical system

$$\frac{\partial}{\partial x} z(x,t) = J\left(-H_1(x)\frac{\partial}{\partial t} z(x,t) + H_0(x) z(x,t)\right). \tag{7.1}$$

In order to construct Darboux transformation of system (7.1) and solutions of the transformed system, we use (2.17) and rewrite (2.10) (for our case where the relations (2.13)–(2.15) are valid) in the form

$$\big(J\Pi^* S^{-1}\big)' = J\big(H_1 J\Pi^* S^{-1} A + \widetilde{H}_0 J\Pi^* S^{-1}\big), \tag{7.2}$$

$$\widetilde{H}_0 = H_0 - X^* H_1 - H_1 X, \quad X = J\Pi^* S^{-1}\Pi. \tag{7.3}$$

We note that (7.3) is equivalent to the second equality in (2.20).

Proposition 7.1. *Let J, $H_1(x)$ and $H_0(x)$ satisfying (2.14), as well as the triple $\{A, S(0) = S(0)^*, \Pi(0)\}$ satisfying (3.10), be given. Let the matrix functions $\Pi(x)$ and $S(x)$ be determined by (2.16). Then the vector functions*

$$\widetilde{z}(x,t) = J\Pi(x)^* S(x)^{-1} \mathrm{e}^{-tA} h \quad (h \in \mathbb{C}^m) \tag{7.4}$$

satisfy, in the points of invertibility of $S(x)$, the transformed dynamical system (of the same form as (7.1)). More precisely, we have

$$\frac{\partial}{\partial x}\widetilde{z}(x,t) = J\left(-H_1(x)\frac{\partial}{\partial t}\widetilde{z}(x,t) + \widetilde{H}_0(x)\widetilde{z}(x,t)\right), \tag{7.5}$$

where $\widetilde{H}_0$ is given by (7.3)

Proof. In view of (7.2) and (7.4), both sides of (7.5) equal $J\big(H_1 J\Pi^* S^{-1}A + \widetilde{H}_0 J\Pi^* S^{-1}\big)\mathrm{e}^{-tA}h$. □

When $H_1 \geq 0$, the energy $E_z(t)$ of the solutions z of system (7.1) on $[0,a]$ $(0 < a < \ell)$ is given by the formula

$$E_z(t)^2 = \int_0^a z(x,t)^* H_1(x) z(x,t)dx. \tag{7.6}$$

The energy of the transformed solutions $\widetilde{z}$ of the form (7.4) is expressed via A and $S(x)$.

Proposition 7.2. *Let the conditions of Proposition 7.1 hold and assume additionally that $H_1 \geq 0$ and $S(0) > 0$. Then the energy $E_{\widetilde{z}}$, where $\widetilde{z}$ has the form (7.4), is given by the formula*

$$E_{\widetilde{z}}(t) = \sqrt{h^*\mathrm{e}^{-tA^*}\big(S(0)^{-1} - S(a)^{-1}\big)\mathrm{e}^{-tA}h}. \tag{7.7}$$

Proof. Taking into account (5.8) and (7.4) we see that

$$\widetilde{z}(x,t)^* H_1(x)\widetilde{z}(x,t) = -h^*\mathrm{e}^{-tA^*}\big(S(x)^{-1}\big)'\mathrm{e}^{-tA}h. \tag{7.8}$$

Formula (7.7) follows from (7.6) and (7.8). □

7.2. Two-Way Diffusion Equation

In this subsection, we consider the important case when J, H_1 and H_0 have the form (5.4) (i.e., the same form as in Lagrange-symmetric Shin–Zettl system) and $\omega = \overline{\omega}$, $p = \overline{p}$, $q = \overline{q}$. In that case we set

$$z(x,t) = \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix}, \quad \widetilde{z}(x,t) = \begin{bmatrix} \widetilde{z}_1(x,t) \\ \widetilde{z}_2(x,t) \end{bmatrix}, \tag{7.9}$$

and rewrite (7.1) in the form

$$z_1' = rz_1 + p^{-1}z_2, \quad z_2' = \omega\frac{\partial}{\partial t}z_1 + qz_1 - \overline{r}z_2 \quad \left(z_i' = \frac{\partial}{\partial x}z_i\right). \tag{7.10}$$

Next, we rewrite the first equality in (7.10) as $z_2 = p(z_1' - rz_1)$, substitute the expression for z_2 into the second equality in (7.10) and obtain

$$\omega\frac{\partial}{\partial t}z_1 = \big(p(z_1' - rz_1)\big)' - qz_1 + \overline{r}p(z_1' - rz_1). \tag{7.11}$$

In particular, when $r = 0$, Eq. (7.11) takes the form

$$\omega\frac{\partial}{\partial t}z_1 = \big(p(z_1')\big)' - qz_1. \tag{7.12}$$

We note that Eq. (7.12) coincides (in the case of sign-indefinite ω) with the *two-way diffusion equation* (6.1) in [28]. See also various references in [4,16,28] on the literature related to the two-way diffusion equation.

According to Corollary 5.1, $\widetilde{H}_0$ has the same form as H_0. More precisely, we have (see (5.10) or (7.3)):

$$\widetilde{H}_0(x) = \begin{bmatrix} -\widetilde{q}(x) & \overline{\widetilde{r}(x)} \\ \widetilde{r}(x) & p(x)^{-1} \end{bmatrix}, \quad \widetilde{r} = r - \omega X_{12}, \quad \widetilde{q} = q + \omega(X_{11} + \overline{X_{11}}). \quad (7.13)$$

In the same way as (7.1) yields (7.11), Eq. (7.5) implies that the entry $\widetilde{z}_1$ of the solution $\widetilde{z}$ given by (7.4) satisfies the equation

$$\omega \frac{\partial}{\partial t}\widetilde{z}_1 = \big(p(\widetilde{z}_1' - \widetilde{r}\widetilde{z}_1)\big)' - \widetilde{q}\widetilde{z}_1 + \overline{\widetilde{r}}p(\widetilde{z}_1' - \widetilde{r}\widetilde{z}_1). \quad (7.14)$$

Assuming $r \equiv 0$, we see that

$$\widetilde{r} = -\omega X_{12}, \quad \widetilde{q} = q + \omega(X_{11} + \overline{X_{11}}). \quad (7.15)$$

Multiplying the left-hand side of (6.6) by "-1" and substituting there $\widetilde{y}_1 = \widetilde{z}_1$ we obtain the right-hand side of (7.14). Hence, the proof of Proposition of 6.1 shows that

$$\big(p(\widetilde{z}_1' - \widetilde{r}\widetilde{z}_1)\big)' - \widetilde{q}\widetilde{z}_1 + \overline{\widetilde{r}}p(\widetilde{z}_1' - \widetilde{r}\widetilde{z}_1) = \big(p\widetilde{z}_1'\big)' - \breve{q}\widetilde{z}_1, \quad (7.16)$$

$$\breve{q} = q + 2\omega(X_{11} - X_{22}) + 2p(\omega X_{12})^2 - (p\omega)'X_{12}. \quad (7.17)$$

From (7.9), (7.14) and (7.16), the next proposition is immediate.

Proposition 7.3. *Let J, H_1 and H_0 have the form (5.4), and let the function $p\omega$ be differentiable and its derivative $(p\omega)'$ as well as the functions p^{-1}, q and ω be locally summable on $[0, \ell)$. Assume that (6.1) holds and that the triple $\{A, S(0) = S(0)^*, \Pi(0)\}$ satisfies (3.10). Introduce $\Pi(x)$ and $S(x)$ via (2.16).*

Then the function $\widetilde{z}_1$ (given by (7.4) and (2.16)) satisfies, in the points of invertibility of $S(x)$, the dynamical equation

$$\omega \frac{\partial}{\partial t}\widetilde{z}_1 = \big(p\widetilde{z}_1'\big)' - \breve{q}\widetilde{z}_1, \quad (7.18)$$

where $\breve{q}$ is given by (7.17).

Recall that (7.18) is an equation of the form (7.12).

Remark 7.4. It is important that [46, Theorem 7.4] and our Theorem 2.1, in particular, is valid on any interval $\mathcal{I}$ such that $0 \in \mathcal{I}$. Thus, the previous statements of the paper, excluding the last sentence in Proposition 2.2, Remark 2.3, Proposition 7.2 and the statements from Sect. 3 (where the condition $S(x) > 0$ is essential), are also valid on the intervals $\mathcal{I}$ such that $0 \in \mathcal{I}$. The interval $[0, \ell)$ was chosen for simplicity but the interval $(-\ell, \ell)$ is sometimes more convenient in the two-way diffusion equation and in the indefinite Sturm–Liouville case.

8. Transformed Systems: Inserted Eigenvalues

The insertion of real eigenvalues into the spectrum of Schrödinger and Dirac operators was actively studied using Crum–Krein and commutation methods (see, e.g., important papers [10,17,32,52]). In the case of none-self-adjoint and *indefinite* first order differential systems and *indefinite* Sturm–Liouville equations, non-real eigenvalues are of interest (see [6,7,11,26,27,41] and references therein). Thus, the insertion (and the removal as an operation which is inverse to the insertion) of non-real eigenvalues is of interest as well. Here (in this section and in Sect. 9), we begin this study and consider real and non-real eigenvalues inserted via the parameter matrix A.

In the GBDT approach, the Darboux matrix w_A of the form (2.5) or (2.19) is basic for the construction of fundamental solutions of the transformed systems (in the case of the so called spectral systems with a spectral parameter). On the other hand, when we apply GBDT to the construction of explicit solutions of dynamical systems (as in Sect. 7) or to the construction of eigenfunctions and generalised eigenfunctions (in this section), we use relations (2.10) or (7.2).

When J and $H_1(x)$ are invertible, we introduce differential expressions

$$\mathcal{L} := H_1(x)^{-1}\left(J^{-1}\frac{d}{dx} - H_0(x)\right), \quad \widetilde{\mathcal{L}} := H_1(x)^{-1}\left(J^{-1}\frac{d}{dx} - \widetilde{H}_0(x)\right), \tag{8.1}$$

and rewrite systems (1.1) and (2.22) in the forms $\mathcal{L}y = \lambda y$ and $\widetilde{\mathcal{L}}\widetilde{y} = \lambda\widetilde{y}$, respectively. In view of (7.2), we obtain the following proposition.

Proposition 8.1. *Let* (2.14) *hold, let J and $H_1(x)$ be invertible and let the transformed matrix function $\widetilde{H}_0$ be given by the relations* (2.20), (2.21) *using the triple of parameter matrices $\{A, S(0) = S(0)^*, \Pi(0)\}$ satisfying*

$$AS(0) - S(0)A^* = \Pi(0)J\Pi(0)^*.$$

Moreover, let $\widetilde{L}$ be a linear operator generated by the differential expression $\widetilde{\mathcal{L}}$ on $(-\ell, \ell)$ or on $[0, \ell)$, $\ell \le \infty$; let the columns of $J\Pi^*S^{-1}$ belong to the domain of $\widetilde{L}$, and assume that f is a generalised eigenvector of the matrix A (more precisely, an eigenvector corresponding to the eigenvalue μ and of rank $k \ge 1$). Assume that the columns of $J\Pi(x)^*S(x)^{-1}$ are linearly independent vector functions.

Then, $J\Pi(x)^*S(x)^{-1}f$ is a generalised vector eigenfunction of rank k, which corresponds to the eigenvalue μ of the operator $\widetilde{L}$.

Proof. The intertwining relation

$$(\widetilde{L} - \mu I)J\Pi^*S^{-1}g = J\Pi^*S^{-1}(A - \mu I_n)g \quad \text{for all } g \in \mathbb{C}^n \tag{8.2}$$

is immediate from (7.2). The statement of the proposition follows. □

The remark below is useful when the invertibility of $S(x)$ and the condition (from Proposition 8.1) that the columns of $J\Pi^*S^{-1}$ belong to the domain of $\widetilde{L}$ are studied.

Remark 8.2. Assume that $H_1(x) \ge \varepsilon I_m > 0$ for $x \ge 0$, that $S(0) > 0$, and let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be considered on $[0, \infty)$. Then $S(x)$ is invertible and the columns of $J\Pi^*S^{-1}$ belong to $L^2_m(0, \infty)$. Indeed, since $S' = \Pi J^* H_1 J\Pi^*$ (see (2.16) and take into account that $J^* = -J$), we have $S(x) > 0$ for $x \ge 0$, and

$$\begin{aligned}&\int_0^\infty S(x)^{-1}\Pi(x)J^*J\Pi(x)^*S(x)^{-1}dx\\&\quad\leq(1/\varepsilon)\int_0^\infty S(x)^{-1}\Pi(x)J^*H_1(x)J\Pi(x)^*S(x)^{-1}dx\\&\quad=-(1/\varepsilon)\int_0^\infty\big(S(x)^{-1}\big)'dx\leq(1/\varepsilon)S(0)^{-1}.\end{aligned}\tag{8.3}$$

In the indefinite case, we may achieve that $S(x)$ is invertible for each $x\in(-\infty,\infty)$. For instance, assuming that $H_1(x)>0$ for $x>0$, $H_1(x)<0$ for $x<0$ and $S(0)>0$, we have $S(x)>0$.

In order to study the insertion of eigenvalues in the case of Sturm–Liouville operators, we consider a Lagrange-symmetric Shin–Zettl system and partition $J\Pi^*S^{-1}$ into two rows, that is, we set

$$J\Pi^*S^{-1}=\Omega=\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}.\tag{8.4}$$

Using (5.4) and (7.13), and assuming that $r\equiv 0$, we rewrite (7.2) in the form

$$p\Omega_1'=p\widetilde{r}\Omega_1+\Omega_2,\quad \Omega_2'=-\omega\Omega_1A+\widetilde{q}\Omega_1-\widetilde{r}\Omega_2;\tag{8.5}$$

$$\widetilde{r}=-\omega X_{12},\quad \widetilde{q}=q+\omega(X_{11}-X_{22}).\tag{8.6}$$

Here we also used the equalities $X_{12}=\overline{X_{12}}$ (i.e., $\widetilde{r}=\overline{\widetilde{r}}$) and $\overline{X_{11}}=-X_{22}$, which are immediate from the fact that X_{ij} are the blocks of $X=J\Pi^*S^{-1}\Pi$. We proceed similar to the proof of (7.18). Namely, differentiating the first equation in (8.5) and taking into account the second equation in (8.5) as well as relations (6.9) and (8.6), we derive

$$-\big(p\Omega_1'\big)'+\breve{q}\Omega_1=\omega\Omega_1A,\tag{8.7}$$

where $\breve{q}$ is given by (7.17). When $\omega\neq 0$, we introduce differential expressions

$$\mathcal{K}u=\frac{1}{\omega}\big(-\big(pu'\big)'+qu\big),\quad \widetilde{\mathcal{K}}\widetilde{u}=\frac{1}{\omega}\big(-\big(p\widetilde{u}'\big)'+\breve{q}\widetilde{u}\big).\tag{8.8}$$

From (8.7), quite similarly to the proof of Proposition 8.1 follows the next proposition.

Proposition 8.3. *Let the functions ω, p and q satisfy both the equalities (6.1) and the inequality $\omega\neq 0$ on $(-\ell,\ell)$ or on $[0,\ell)$, $\ell\leq\infty$; let the function $p\omega$ be differentiable and its derivative $(p\omega)'$ as well as the functions p^{-1}, q and ω be locally summable. Assume that the triple $\{A, S(0)=S(0)^*, \Pi(0)\}$ satisfies (3.10). Introduce $\Pi(x)$ and $S(x)$ via (2.16), where J, H_1 and H_0 have the form (5.4) and $r\equiv 0$, and let $S(x)$ be invertible.*

Moreover, let $\widetilde{K}$ be a linear operator generated by the differential expression $\widetilde{\mathcal{K}}$ on $(-\ell,\ell)$ or on $[0,\ell)$, let the entries of $\Omega_1=\begin{bmatrix}0 & 1\end{bmatrix}\Pi^*S^{-1}$ be linearly independent functions, which belong to the domain of $\widetilde{K}$, and assume that f is a generalised eigenvector of the matrix A (more precisely, an eigenvector corresponding to the eigenvalue μ and of rank $k\geq 1$).

Then, $\Omega_1(x)f$ is a generalised eigenfunction of rank k, which corresponds to the eigenvalue μ of the operator $\widetilde{K}$.

9. Indefinite Sturm–Liouville Equations

First order systems, where H_1 is not necessarily nonnegative, and indefinite Sturm–Liouville equations are of growing interest in the literature (see, e.g., [3,7,8,33,34,40] and references therein). Therefore, in this section of the paper we will consider some examples of Darboux transformation for the Lagrange-symmetric Shin–Zettl system and indefinite Sturm–Liouville equation on the real axis $(-\infty, \infty)$, see Remark 7.4.

More precisely, we will construct explicit solutions and generalised eigenfunctions for the interesting model case

$$\omega(x) = \mathrm{sgn}(x), \quad p(x) \equiv 1. \tag{9.1}$$

(Assumptions (9.1) were made, for instance, in [26].)

First, we consider Shin–Zettl system (1.1), (5.4) and assume that the equalities (9.1) and

$$q(x) = r(x) \equiv 0 \tag{9.2}$$

hold for the initial system. Partitioning $\Pi(x)$ into the two columns $\Pi(x) = \big[\Lambda_1(x)\ \Lambda_2(x)\big]$ and taking into account (9.1) and (9.2), we rewrite the first system in (5.8) in the form

$$\Lambda_1' = \begin{cases} A\Lambda_2 & \text{for } x > 0 \\ -A\Lambda_2 & \text{for } x < 0 \end{cases}; \qquad \Lambda_2' = -\Lambda_1. \tag{9.3}$$

Let us set $\Lambda_k(0) = g_k$ (i.e., $\Pi(0) = \big[g_1\ g_2\big]$) and assume that

$$A = \alpha^2 \quad (\det \alpha \neq 0), \quad S(0) = I_n. \tag{9.4}$$

It is immediate that the vector functions

$$\begin{matrix} \Lambda_1(x) = \big(\mathrm{e}^{\mathrm{i}x\alpha}(g_1 - \mathrm{i}\alpha g_2) + \mathrm{e}^{-\mathrm{i}x\alpha}(g_1 + \mathrm{i}\alpha g_2)\big)/2 \\ \Lambda_2(x) = \mathrm{i}\alpha^{-1}\big(\mathrm{e}^{\mathrm{i}x\alpha}(g_1 - \mathrm{i}\alpha g_2) - \mathrm{e}^{-\mathrm{i}x\alpha}(g_1 + \mathrm{i}\alpha g_2)\big)/2 \end{matrix} \quad \text{for } x \geq 0; \tag{9.5}$$

$$\begin{matrix} \Lambda_1(x) = \big(\mathrm{e}^{x\alpha}(g_1 - \alpha g_2) + \mathrm{e}^{-x\alpha}(g_1 + \alpha g_2)\big)/2 \\ \Lambda_2(x) = -\alpha^{-1}\big(\mathrm{e}^{x\alpha}(g_1 - \alpha g_2) - \mathrm{e}^{-x\alpha}(g_1 + \alpha g_2)\big)/2 \end{matrix} \quad \text{for } x \leq 0 \tag{9.6}$$

satisfy both (9.3) and the initial conditions $\Lambda_k(0) = g_k$. The second system in (5.8) takes the form $S' = \omega\Lambda_2\Lambda_2^*$, that is,

$$S(x) = S(0) + \int_0^x \Lambda_2(t)\Lambda_2(t)^* dt \geq S(0) \quad (x > 0), \tag{9.7}$$

$$S(x) = S(0) + \int_x^0 \Lambda_2(t)\Lambda_2(t)^* dt \geq S(0) \quad (x < 0). \tag{9.8}$$

Formulas (9.5)–(9.8) present explicit expressions for $\Pi(x)$ and $S(x)$, and so the Darboux matrix $w_A(x, \lambda)$ of the form (5.9) is constructed explicitly.

We note that (in view of (9.4)), the matrix identity (3.10), which should hold for the triple $\{A, S(0), \Pi(0)\}$, takes the form

$$\alpha^2 - (\alpha^*)^2 = g_1 g_2^* - g_2 g_1^*. \tag{9.9}$$

In order to use Corollary 5.1, we also solve explicitly the initial Shin–Zettl system (1.1), (5.4), where (9.1) and (9.2) hold. Namely, we introduce matrices

$$T_+(\lambda)=\begin{bmatrix}1 & 1\\ \mathrm{i}\sqrt{\lambda} & -\mathrm{i}\sqrt{\lambda}\end{bmatrix}, \quad D_+(\lambda)=\begin{bmatrix}\mathrm{i}\sqrt{\lambda} & 0\\ 0 & -\mathrm{i}\sqrt{\lambda}\end{bmatrix}, \tag{9.10}$$

$$T_-(\lambda)=\begin{bmatrix}1 & 1\\ \sqrt{\lambda} & -\sqrt{\lambda}\end{bmatrix}, \quad D_-(\lambda)=\begin{bmatrix}\sqrt{\lambda} & 0\\ 0 & -\sqrt{\lambda}\end{bmatrix}, \tag{9.11}$$

where $\sqrt{\lambda}$ is any fixed branch of the square root of λ. It is easy to see that (in our case) F given in (1.1) satisfies the equalities $FT_+=T_+D_+$ for $x>0$ and $FT_-=T_-D_-$ for $x<0$. Therefore, solutions y of the initial Shin–Zettl system (1.1) are given by the formulas

$$y(x,\lambda)=T_+(\lambda)\mathrm{e}^{xD_+(\lambda)}T_+(\lambda)^{-1}h \quad (x>0), \tag{9.12}$$

$$y(x,\lambda)=T_-(\lambda)\mathrm{e}^{xD_-(\lambda)}T_-(\lambda)^{-1}h \quad (x<0) \tag{9.13}$$

with any vectors $h\in\mathbb{C}^2$. Now, Corollary 5.1 yields our next corollary.

Corollary 9.1. *Assume that the initial Shin–Zettl system on* $(-\infty,\infty)$ *has the form* (1.1), (5.4) *and that equalities* (9.1) *and* (9.2) *are valid. Let the matrices* A *and* $S(0)$ *have the form* (9.4), *and let* (9.9) *hold.*

Then, $S(x)$ *is invertible, and the corresponding GBDT-transformed Shin-Zettl system is well defined and is given explicitly by the formulas*

$$\widetilde{y}'(x,\lambda)=J(\lambda H_1(x)+\widetilde{H}_0(x))\widetilde{y}(x,\lambda), \quad \widetilde{H}_0(x)=\begin{bmatrix}-\widetilde{q}(x) & \widetilde{r}(x)\\ \widetilde{r}(x) & 1\end{bmatrix}, \tag{9.14}$$

$$\widetilde{r}(x)=\overline{\widetilde{r}(x)}=-\mathrm{sgn}(x)X_{12}(x), \quad \widetilde{q}(x)=\mathrm{sgn}(x)(X_{11}(x)+\overline{X_{11}(x)}), \tag{9.15}$$

where X_{ij} *are the blocks of* $X=J\Pi^*S^{-1}\Pi$, *and explicit expressions for* S *and* Π *are given in* (9.5)–(9.8).

Moreover, system (9.14) *is Lagrange symmetric, and its solutions* $\widetilde{y}$ *are explicitly expressed via the formula* $\widetilde{y}(x,\lambda)=w_A(x,\lambda)y(x,\lambda)$, *where* y *is given by* (9.12) *and* (9.13).

By virtue of Proposition 6.1, Remark 7.4 and Corollary 9.1, we obtain explicit solutions of indefinite Sturm–Liouville systems

$$-\widetilde{y}_1''(x,\lambda)+\breve{q}(x)\widetilde{y}_1(x,\lambda)=\lambda\,\mathrm{sgn}(x)\widetilde{y}_1(x,\lambda) \quad (-\infty<x<\infty). \tag{9.16}$$

Corollary 9.2. *Assume that relations* (9.4) *and* (9.9) *hold, and let* $\Pi(x)$ *and* $S(x)$ *be given by* (9.5)–(9.8) . *Set* $\widetilde{y}(x,\lambda)=w_A(x,\lambda)y(x,\lambda)$, *where explicit expressions for* $w_A(x,\lambda)$ *and* $y(x,\lambda)$ *are given by* (5.9) *and* (9.12), (9.13), *respectively. Then the first entry* $\widetilde{y}_1$ *of* $\widetilde{y}$ *satisfies the indefinite Sturm–Liouville system* (9.16) *where*

$$\breve{q}(x)=2\,\mathrm{sgn}(x)\big(X_{11}(x)-X_{22}(x)\big)+2X_{12}(x)^2 \tag{9.17}$$

and X_{ij} *are the blocks of* $X=J\Pi^*S^{-1}\Pi$.

Recall that the operator $\widetilde{K}$ was introduced and its generalised eigenfunctions $\Omega_1 f$ $(\Omega_1=\Lambda_2^*S^{-1})$ were considered in Proposition 8.3. Now, we will study $\widetilde{K}$ generated by the differential expression $\widetilde{\mathcal{K}}$, which takes the form

$$\widetilde{\mathcal{K}}\widetilde{u}=\mathrm{sgn}(x)\big(-\widetilde{u}''+\breve{q}\widetilde{u}\big) \tag{9.18}$$

on the domain $\mathcal{D}$ of the absolutely continuous functions $\widetilde{u}$ from $L^2(-\infty,\infty)$, such that $\widetilde{u}'(x)$ are absolutely continuous (for $x\neq 0$) and $\widetilde{\mathcal{K}}\widetilde{u}\in L^2(-\infty,\infty)$.

We note that it is often required that $\widetilde{u}$ (from $\mathcal{D}$) is absolutely continuous everywhere on $(-\infty, \infty)$ (see [26]), although the requirements in [7] are somewhat weaker. Proposition 8.3 yields the following corollary.

Corollary 9.3. *Let the operator $\widetilde{K}$ be generated by the differential expression $\widetilde{\mathcal{K}}$ of the form (9.18), where $\breve{q}$ is given by (9.17), $X = J\Pi^* S^{-1}\Pi$, $\Pi = \begin{bmatrix}\Lambda_1 \ \Lambda_2\end{bmatrix}$, Λ_k $(k = 1, 2)$ are given by (9.5) and (9.6), and S is given by (9.7) and (9.8). Assume that (9.4) and (9.9) hold. Moreover, assume that the entries of $\Lambda_2^* S^{-1}$ are linearly independent functions and that f is a generalised eigenvector of the matrix A (more precisely, an eigenvector corresponding to the eigenvalue μ and of rank $k \geq 1$).*

Then, $\Lambda_2(x)^* S(x)^{-1} f$ is a generalised eigenfunction of rank k, which corresponds to the eigenvalue μ of the operator $\widetilde{K}$.

Proof. In order to show that the conditions of the Proposition 8.3 are fulfilled (and to prove our corollary in this way), it remains only to show that the entries of $\Lambda_2^* S^{-1}$ are squarely summable.

Indeed, we have $S'(x) = \mathrm{sgn}(x)\Lambda_2(x)\Lambda_2(x)^*$, which implies that

$$S(x)^{-1}\Lambda_2(x)\Lambda_2(x)^* S(x)^{-1} = -\mathrm{sgn}(x)\big(S(x)^{-1}\big)'. \tag{9.19}$$

From (9.19), it is immediate that

$$\int_{-\infty}^{\infty} S(x)^{-1}\Lambda_2(x)\Lambda_2(x)^* S(x)^{-1} dx \leq 2S(0)^{-1} = 2I_n. \tag{9.20}$$

Thus, $\Lambda_2^* S^{-1}$ is squarely summable, and so its entries belong to the domain $\mathcal{D}$ of $\widetilde{\mathcal{K}}$. The conditions of the Proposition 8.3 are fulfilled. □

Remark 9.4. In connection with Corollary 9.3, we note that (9.9) holds for $\alpha = \alpha_0 + \mathrm{i}\, c\, hh^*$, where $\alpha_0 = \alpha_0^*$, $c = \overline{c}$, $h \in \mathbb{C}^n$ and we set $g_1 = \mathrm{i}\alpha_0 h$, $g_2 = 2c\, h$. In this case, it would be of interest to study how the choice of α_0 and h influences the Jordan structure of $A = \alpha^2$.

Remark 9.5. Interesting indefinite Sturm-Liouville equations with the potentials $\breve{q}(x)$, which are singular at $x = 0$, are generated by the triples of matrices $\{A,\, S(0),\, \Pi(0)\}$ of the form

$$A = \alpha^2, \quad S(0) = 0, \quad \Pi(0) = \begin{bmatrix}-2\mathrm{i}\alpha g & 2\mu\alpha g\end{bmatrix}, \tag{9.21}$$

where α are $n \times n$ matrices, $g \in \mathbb{C}^n$ are vector columns, μ are purely imaginary values (i.e., $\overline{\mu} = -\mu$), and

$$\det(\mu\alpha \pm I_n) \neq 0, \quad \det(\mu\alpha \pm \mathrm{i}I_n) \neq 0. \tag{9.22}$$

It is easily checked that the third equality in (9.21) yields $\Pi(0)J\Pi(0)^* = 0$ $(J = \mathrm{i}\sigma_2)$, and so the matrix identity (3.10), which is required in GBDT, holds for the triple of the form (9.21). We note that $S(x) > 0$ for $x \neq 0$ if the pair $\widehat{\alpha} := \begin{bmatrix}\alpha & 0 \\ 0 & -\alpha\end{bmatrix}$ and $\widehat{g} := \begin{bmatrix} g \\ g\end{bmatrix}$ is controllable. That is, in this case the singularity is restricted to the point $x = 0$. Some particular cases (but in greater detail) were considered in [29, Section 5], and it was proved for those cases that $\breve{q}(x) = O(x^{-2})$ when x tends to 0.

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