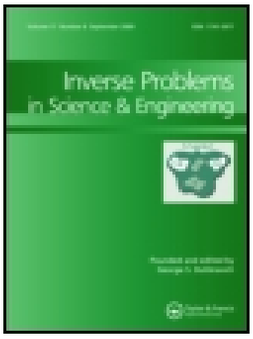

Inverse Problems in Science and Engineering

ISSN: 1741-5977 (Print) 1741-5985 (Online) Journal homepage: http://www.tandfonline.com/loi/gipe20

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To cite this article: Christian Gerhards (2018): On the reconstruction of inducing dipole directions and susceptibilities from knowledge of the magnetic field on a sphere, Inverse Problems in Science and Engineering, DOI: 10.1080/17415977.2018.1438426

To link to this article: https://doi.org/10.1080/17415977.2018.1438426

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INVERSE PROBLEMS IN SCIENCE AND ENGINEERING, 2018
https://doi.org/10.1080/17415977.2018.1438426

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On the reconstruction of inducing dipole directions and susceptibilities from knowledge of the magnetic field on a sphere

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ABSTRACT

Reconstructing magnetizations from measurements of the generated magnetic potential is generally non-unique. The non-uniqueness still remains if one restricts the magnetization to those induced by an ambient magnetic dipole field (i.e. the magnetization is described by a scalar susceptibility and the dipole direction). Here, we investigate the situation under the additional constraint that the susceptibility is either spatially localized in a subregion of the sphere or that it is band-limited. If the dipole direction is known, then the susceptibility is uniquely determined under the spatial localization constraint while it is only determined up to a constant under the assumption of band-limitedness. If the dipole direction is not known, uniqueness is lost again. However, we show that all dipole directions that could possibly generate the measured magnetic potential need to be zeros of a certain polynomial which can be computed from the given potential. We provide examples of non-uniqueness of the dipole direction and examples on how to find admissible candidates for the dipole direction under the spatial localization constraint.

ARTICLE HISTORY
Received 20 July 2017
Accepted 22 January 2018

KEYWORDS
Inverse magnetization problem; decomposition of spherical vector fields; uniqueness; magnetic dipoles; susceptibility

AMS SUBJECT CLASSIFICATIONS
45Q05; 33C55; 86A22; 86A25

1. Introduction

Assuming a magnetic field $\mathbf{B}$ of the form $\mathbf{B} = \nabla V$ on a sphere $\mathbb{S}_R = \{x \in \mathbb{R}^3 : |x| = R\}$ that is generated by a magnetization $\mathbf{m}$ on a sphere $\mathbb{S}_r$ of radius $r < R$, we are interested in the question of which contributions of $\mathbf{m}$ can be reconstructed from knowledge of the potential V. In particular, we are interested in magnetizations that are induced by an ambient dipole magnetic field, i.e. $\mathbf{m}$ is of the form

$$\mathbf{m}(x) = Q(x)\frac{3(x \cdot \mathbf{d})x - \mathbf{d}|x|^2}{|x|^5}, \quad x \in \mathbb{S}_r, \tag{1.1}$$

where $\mathbf{d} \in \mathbb{S}$ denotes the direction of the dipole (which we assume to be located in the center of the sphere $\mathbb{S}_r$) and Q the susceptibility on $\mathbb{S}_r$. For brevity, several physical quantities and constants are implicitly included in the function Q (so, technically, Q is not a susceptibility,

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but we still call it 'susceptibility' throughout this paper). More precisely, $Q(x) = \chi(x)\frac{d}{4\pi}$, where χ is the actual susceptibility and d the norm of the inducing dipole moment. For the magnetic field $\mathbf{B}$ we assume that it has no other sources than $\mathbf{m}$, i.e. outside $\mathbb{S}_r$, it can be written in the form $\mathbf{B} = \nabla V$ with a harmonic potential V given by

$$V(x) = V[\mathbf{m}](x) = \frac{1}{4\pi}\int_{\mathbb{S}_r} \mathbf{m}(y)\cdot\frac{x-y}{|x-y|^3}\mathrm{d}\omega(y), \quad x \in \mathbb{R}^3 \setminus \mathbb{S}_r. \tag{1.2}$$

When $\mathbf{m}$ is of induced form as described in (1.1), we typically write $V[Q, \mathbf{d}]$ instead of the more general notation $V[\mathbf{m}]$.

In general, even if the dipole direction $\mathbf{d}$ is known, the susceptibility Q is not determined uniquely by knowledge of the potential $V[Q, \mathbf{d}]$ on the sphere $\mathbb{S}_R$ (see, e.g. [1], where they named magnetizations that produce no magnetic potential on $\mathbb{S}_R$ 'annihilators'; here, we call such magnetizations 'silent from outside'). If we make the additional assumption that Q is locally supported in some subregion $\Gamma \subset \mathbb{S}_r$, then the susceptibility is actually determined uniquely (cf. [2], based on results from [3,4] in a Euclidean set-up). Therefore, in the latter scenario, but under the condition that the dipole direction is not known, our goal is to find suitable candidates for the dipole direction $\mathbf{d}$. If the magnetization $\mathbf{m}$ was known, then a standard procedure such as described in [5, Chapter 7] can be used to derive $\mathbf{d}$ from the direction of $\mathbf{m}$ or to see that $\mathbf{m}$ cannot be of the form (1.1). However, just given the corresponding magnetic potential $V[\mathbf{m}]$ on the sphere $\mathbb{S}_R$, only certain components of $\mathbf{m}$ can be reconstructed uniquely (cf. [2,3,6]; a summary is provided in Section 2). Namely, if

$$\mathbf{m} = \tilde{\mathbf{m}}^{(1)} + \tilde{\mathbf{m}}^{(2)} + \tilde{\mathbf{m}}^{(3)} \tag{1.3}$$

is the Hardy-Hodge decomposition (cf. Theorem 2.2) of the magnetization, then only $\tilde{\mathbf{m}}^{(2)}$ is determined uniquely. If we know in advance that $\mathbf{m}$ is localized in a subregion $\Gamma \subset \mathbb{S}_r$, then $\tilde{\mathbf{m}}^{(1)}$ and $\tilde{\mathbf{m}}^{(2)}$ are determined uniquely.

In other words, the question we are interested in can be reformulated as follows: Knowing only the uniquely determined components of $\mathbf{m}$, what can be said about $\mathbf{d}$ and Q? An illustration of the effect of non-uniqueness on classical methods of paleopole estimation can be found in [7], for the case that no further a priori assumptions are made on the induced magnetization. In the paper at hand, we investigate the influence of certain constraints on $\mathbf{m}$ (namely, the constraint that the magnetization is localized in a subdomain $\Gamma \subset \mathbb{S}_r$ of the sphere or that it is band-limited). More precisely, we provide examples of non-uniqueness for the simultaneous reconstruction of Q and $\mathbf{d}$ from knowledge of $V[Q, \mathbf{d}]$ on $\mathbb{S}_R$, even under the mentioned additional constraints. But we also show that all possible candidates for the dipole direction $\mathbf{d}$ for which the given potential can be expressed in the form $V[Q, \mathbf{d}]$ need to be zeros of a particular polynomial that can be obtained from the given potential (cf. Sections 3 and 4). This allows to restrict the set of candidates for the dipole direction and, to some extent, improve the handling of the non-uniqueness. An overview on the mentioned uniqueness results for the reconstruction of general or induced magnetizations is provided in Table 1.

The spatial localization constraint can be a geophysically reasonable assumption in certain situations. For example, localization could be enforced if one has knowledge of the true magnetization in a small subregion of $\mathbb{S}_r$ (the magnetic field produced by this known

magnetization can be subtracted from the given magnetic field, so that the residual has to be produced by a magnetization that satisfies the spatial localization assumption) or if it is known in advance that there exists a region with nearly vanishing magnetization (which is expected for some craters on Mars; see, e.g. [8]). Being able to compute the set of admissible candidates for the dipole direction could be of use, e.g. for paleopole estimations. The assumption that the magnetization $\mathbf{m}$ is concentrated on a spherical surface $\mathbb{S}_r$ is fairly common in geophysical applications since magnetization typically occurs only in the upper few tens of kilometers of the Earth's lithosphere. Actually, for any 'sufficiently nice' volumetric magnetization in the ball $\mathbb{B}_r = \{x \in \mathbb{R}^3 : |x| < r\}$ there can be found a magnetization concentrated on $\mathbb{S}_r$ that produces the same magnetic potential on $\mathbb{S}_R$, $r < R$, as its volumetric counterpart (see, e.g. [9, Section 3]). For the notion of vertically integrated magnetizations, the reader is referred to [6]. The complicated fact that magnetizations often have remanent and induced contributions are not accounted for in our approach. We require the knowledge of the induced magnetic field and regard all non-induced contributions as 'noise'. Last, it should be noted that the inversion of the magnetic potential $V[\mathbf{m}]$ from (1.2) is closely related to the gravimetric problem (see, e.g. [10,11] and references therein). However, while the gravimetric problem is unique when restricted to harmonic mass densities, the vectorial nature of the inverse magnetization problem causes the described non-uniqueness issues.

Finally, the structure of the paper at hand is as follows: In Section 2, we provide some notations and a brief recapitulation of the spherical Helmholtz and Hardy-Hodge decompositions. The latter classifies those components of the magnetization $\mathbf{m}$ (not necessarily of the form (1.1)) that are determined uniquely by knowledge of $V[\mathbf{m}]$ on $\mathbb{S}_R$. Namely, if $\mathbf{m} = \tilde{\mathbf{m}}^{(1)} + \tilde{\mathbf{m}}^{(2)} + \tilde{\mathbf{m}}^{(3)}$ is the Hardy-Hodge decomposition, then only $\tilde{\mathbf{m}}^{(2)}$ is determined uniquely (e.g. [2,3,6]; we say that $\mathbf{m}$ and $\tilde{\mathbf{m}}^{(2)}$ are 'equivalent from outside'). We also formulate the Helmholtz and Hardy-Hodge decompositions in terms of some well-known vector spherical harmonics, which will be of use for our considerations on band-limited magnetizations. However, it should already be noted that the constraint of $\mathbf{m}$ being band-limited, opposed to being spatially localized, still only yields that $\tilde{\mathbf{m}}^{(2)}$ is determined uniquely by $V[\mathbf{m}]$ on $\mathbb{S}_R$.

Table 1. Overview on uniqueness results for magnetizations, provided that the magnetic potential V is known on $\mathbb{S}_R$, with radius $R > r$. By $\mathbf{m}$ we mean the underlying magnetization, by $\tilde{\mathbf{m}}^{(1)}$, $\tilde{\mathbf{m}}^{(2)}$, $\tilde{\mathbf{m}}^{(3)}$ the contributions of its Hardy-Hodge decomposition (1.3). In the case of induced magnetizations $\mathbf{m}$ of the form (1.1), $\mathbf{d}$ denotes the inducing dipole direction and $Q = \chi \frac{d}{4\pi}$, where χ is the susceptibility and d the norm of the inducing dipole moment.

	General mag.	Induced mag. ($\mathbf{d}$ known)	Induced mag.
No constraint	$\tilde{\mathbf{m}}^{(2)}$ unique $\tilde{\mathbf{m}}^{(1)}, \tilde{\mathbf{m}}^{(3)}$ non-unique (cf. [2,3,6])	Q non-unique (cf. [1,12])	Q, $\mathbf{d}$ non-unique (cf. [1,7,12])
Band-limited	$\tilde{\mathbf{m}}^{(2)}$ unique $\tilde{\mathbf{m}}^{(1)}, \tilde{\mathbf{m}}^{(3)}$ non-unique (cf. [2,3,6])	Q unique up to additive const. (cf. Lemma 4.3)	Q, $\mathbf{d}$ non-unique, candidates for $\mathbf{d}$ (cf. Theorem 4.6)
Spatially localized	$\tilde{\mathbf{m}}^{(1)}, \tilde{\mathbf{m}}^{(2)}$ unique $\tilde{\mathbf{m}}^{(3)}$ non-unique (cf. [2,3])	Q unique (cf. [2,3])	Q, $\mathbf{d}$ non-unique, candidates for $\mathbf{d}$ (cf. Theorem 3.3)

Based on the results from Sections 2, 3 and 4 focus on the case of induced magnetizations of the form (1.1) under the constraint that the susceptibilities Q are localized in a subregion $\Gamma \subset \mathbb{S}_r$ or that Q is band-limited, respectively. In both cases, we supply counter-examples to the uniqueness issue, i.e. we construct two susceptibilities Q and $\overline{Q}$ and dipole directions $\mathbf{d} \neq \pm\overline{\mathbf{d}}$ that satisfy the respective constraints and additionally yield $V[Q, \mathbf{d}] = V[\overline{Q}, \overline{\mathbf{d}}]$ on $\mathbb{S}_R$ (throughout the course of this paper, we call $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ 'equivalent (from outside)' if they produce the same potential on $\mathbb{S}_R$). Although non-uniqueness prevails under the additional constraints, for a given potential V of the form (1.2), we derive a way of computing a subset of $\mathbb{S}$ which contains all dipole directions $\mathbf{d}$ for which there exists a susceptibility Q such that $V = V[Q, \mathbf{d}]$ on $\mathbb{S}_R$. Namely, in the case of spatially localized susceptibilities, the admissible dipole directions are zeros of a fourth order polynomial that can be computed from the known potential V (cf. Theorem 3.3). This way, we at least obtain some additional information on the otherwise non-unique problem. In the optimal case, there exists only a single zero $\pm\mathbf{d}$ of the polynomial, which would guarantee uniqueness for the particular measured magnetic potential V (note that uniqueness is only understood up the sign because, obviously, $V[Q, \mathbf{d}] = V[-Q, -\mathbf{d}]$). Similar results can be obtained for band-limited susceptibilities (cf. Section 4). However, here the degree of the polynomial of which the zeros need to be determined depends on the band-limit (furthermore, the zeros do not directly resemble the dipole direction $\mathbf{d}$ but rather the vector $\mathbf{y}_{\mathbf{d}} = (Y_{1,-1}(\mathbf{d}), Y_{1,0}(\mathbf{d}), Y_{1,1}(\mathbf{d}))$ of spherical harmonics of degree one evaluated at the point $\mathbf{d}$). Additionally, while for the spatial localization constraint, a known dipole direction uniquely determines the susceptibility, the assumption of band-limitedness only implies that a given dipole direction determines the susceptibility up to an additive constant (cf. Lemma 4.3). It should be noted at this point that we included the section on band-limited magnetizations mainly for the sake of completeness, since we believe that the spatial localization constraint could be more useful for geophysically reasonble scenarios. Nonetheless, the band-limited case shows that the commonly used finite expansion of magnetizations in terms of spherical harmonics introduces some level of uniqueness that is not inherent to the original problem (similar to the case of reconstructing magnetic fields from intensity data; see, e.g. [13]).

Eventually, in Section 5, we provide some numerical examples on how the considerations from Section 3 for spatially localized magnetizations can help to obtain suitable candidates for the dipole directions $\mathbf{d}$ and on how to decide if a given potential V on $\mathbb{S}_R$ can be produced by a dipole induced magnetization of the form (1.1) in the first place. For brevity, we restrict the numerical examples to the case of spatial localization constraints (and not the constraint of band-limitation) as we believe this to be more relevant for potential applications. For notational reasons, we choose $r = 1$ throughout the remainder of this paper, since dipole induced magnetizations then have the form

$$\mathbf{m}(x) = Q(x)(3(x \cdot \mathbf{d}) - \mathbf{d}),$$

while the radius R of the sphere where the potential V is given can still be any radius $R > 1$. However, the results hold true for any $0 < r < R$.

2. Auxiliary results and notations

Throughout this paper, bold-face letters $\mathbf{f}, \mathbf{g}$ typically denote vector valued functions mapping $\mathbb{S}$, $\mathbb{S}_R$, or $\mathbb{R}^3$ into $\mathbb{R}^3$, while f, g, F, G denote scalar valued functions mapping $\mathbb{S}$, $\mathbb{S}_R$, or $\mathbb{R}^3$ into $\mathbb{R}$. For brevity, we denote the unit sphere $\mathbb{S}_1 = \{x \in \mathbb{R}^3 : |x| = 1\}$ by $\mathbb{S}$ throughout the course of this paper. Accordingly, $L^2(\mathbb{S}, \mathbb{R}^3)$ and $H_k(\mathbb{S}, \mathbb{R}^3)$ mean the function space of vector valued square-integrable functions and the Sobolev space as denoted, e.g. in [14], respectively. $L^2(\mathbb{S})$ and $H_k(\mathbb{S})$ denote the corresponding scalar valued function spaces. For the rest of this section, we briefly recapitulate some notations and results from [2,3,6,14–17]. In particular, proofs of the main results of this section, i.e. Theorems 2.2, 2.4 and Corollary 2.5, can be found in [2]. First, we define the following Helmholtz operators, acting at a point $x \in \mathbb{S}_r$:

$$o^{(1)} = \frac{x}{|x|}\,\mathrm{id}, \tag{2.1}$$

$$o^{(2)} = \nabla^*, \tag{2.2}$$

$$o^{(3)} = \mathrm{L}^* = \frac{x}{|x|} \times \nabla^*, \tag{2.3}$$

where ∇^* denotes the surface gradient on the unit sphere $\mathbb{S}$, L^* the surface curl gradient ($\times$ means the vector product), and 'id' the identity operator (i.e. the operator $o^{(1)}$ simply represents multiplication with the radial vector). The Euclidean gradient is denoted by ∇ and can be expressed in the form $\nabla = \frac{x}{|x|}\partial_r + \frac{1}{r}\nabla^*$, for $r = |x|$. These operators allow to decompose a spherical vector field into a radial, surface curl-free, and a surface divergence-free tangential contribution.

Theorem 2.1 (Spherical Helmholtz Decomposition): *Any function* $\mathbf{f} \in L^2(\mathbb{S}, \mathbb{R}^3)$ *can be decomposed into*

$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = o^{(1)}[f_1] + o^{(2)}[f_2] + o^{(3)}[f_3], \tag{2.4}$$

where the scalar functions f_1, f_2, f_3 *are uniquely determined by the conditions* $\int_{\mathbb{S}} f_2 \,\mathrm{d}\omega = \int_{\mathbb{S}} f_3 \,\mathrm{d}\omega = 0$.

A further decomposition that is of particular importance for the characterization of magnetizations is based on the spherical Hardy-Hodge operators

$$\tilde{o}^{(1)} = o^{(1)}\left(\mathrm{D} + \frac{1}{2}\right) - o^{(2)}, \tag{2.5}$$

$$\tilde{o}^{(2)} = o^{(1)}\left(\mathrm{D} - \frac{1}{2}\right) + o^{(2)}, \tag{2.6}$$

$$\tilde{o}^{(3)} = o^{(3)}, \tag{2.7}$$

where D denotes the pseudo-differential operator

$$\mathrm{D} = \left(-\Delta^* + \frac{1}{4}\right)^{\frac{1}{2}} \tag{2.8}$$

and $\Delta^* = \nabla^* \cdot \nabla^*$ the spherical Beltrami operator. These operators above reflect the decomposition into a surface curl-free tangential contribution and two further contributions generated by the gradient of functions that are harmonic in the interior and the exterior of $\mathbb{S}$, respectively.

Theorem 2.2 (Spherical Hardy-Hodge Decomposition): *Any function* $\mathbf{f} \in L^2(\mathbb{S}, \mathbb{R}^3)$ *can be decomposed into*

$$\mathbf{f} = \tilde{\mathbf{f}}^{(1)} + \tilde{\mathbf{f}}^{(2)} + \tilde{\mathbf{f}}^{(3)} = \tilde{o}^{(1)}[\tilde{f}_1] + \tilde{o}^{(2)}[\tilde{f}_2] + \tilde{o}^{(3)}[\tilde{f}_3], \tag{2.9}$$

where the scalar functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ *are uniquely determined by the conditions* $\int_{\mathbb{S}} \tilde{f}_1 - \tilde{f}_2 \, \mathrm{d}\omega = \int_{\mathbb{S}} \tilde{f}_3 \, \mathrm{d}\omega = 0$. *If* f_1, f_2, f_3 *are the Helmholtz scalars of* $\mathbf{f}$ *as given in Theorem 2.1, then*

$$\tilde{f}_1 = \frac{1}{2}\left(\mathrm{D}^{-1}[f_1] - f_2 + \frac{1}{2}\mathrm{D}^{-1}[f_2]\right), \tag{2.10}$$

$$\tilde{f}_2 = \frac{1}{2}\left(\mathrm{D}^{-1}[f_1] + f_2 + \frac{1}{2}\mathrm{D}^{-1}[f_2]\right), \tag{2.11}$$

$$\tilde{f}_3 = f_3. \tag{2.12}$$

Although, the Hardy-Hodge decomposition in Theorem 2.2 reflects the decomposition that we require to describe the uniqueness issues of the treated inverse magnetization problem, the contributions f_1, f_2, f_3 from the Helmholtz decomposition in Theorem 2.1 are often easier to handle and compute (e.g. $f_1(x) = \frac{x}{|x|} \cdot \mathbf{f}(x)$). Therefore, the relations (2.10)–(2.12) can be quite helpful. Some related applications and information on such a decomposition on the Euclidean plane instead of a sphere can be found in [3,4,18].

In the following, we recapitulate some earlier results on how the Hardy-Hodge decomposition characterizes the uniqueness of general magnetization $\mathbf{m}$ (for details and proofs, the reader is referred to [2,3]). First, we introduce the notion of equivalent magnetizations, which simply means that the two magnetizations produce the same potential V (i.e. the same magnetic field $\mathbf{B} = \nabla V$) on some sphere $\mathbb{S}_R$. In other words, if there exist two equivalent magnetizations, we have non-uniqueness (i.e. the knowledge of $V = V[\mathbf{m}]$ on $\mathbb{S}_R$ does not uniquely determine $\mathbf{m}$). It should be noted that, when talking about induced magnetizations with susceptibility Q and dipole direction $\mathbf{d} \in \mathbb{S}$, uniqueness is only meant up to the sign because, clearly, $V[Q, \mathbf{d}] = V[-Q, -\mathbf{d}]$.

Definition 2.3: Two magnetizations $\mathbf{m}, \overline{\mathbf{m}} \in L^2(\mathbb{S}, \mathbb{R}^3)$ are called *equivalent from outside* if $V[\mathbf{m}] = V[\overline{\mathbf{m}}]$ on $\mathbb{S}_R$ for an $R > 1$. They are called *equivalent from inside* if $V[\mathbf{m}] = V[\overline{\mathbf{m}}]$ on $\mathbb{S}_R$ for an $R < 1$. A magnetization $\mathbf{m}$ is called *silent from outside* or *inside* if it is equivalent to the zero-magnetization $\overline{\mathbf{m}} \equiv 0$ from outside or inside, respectively (i.e. if $V[\mathbf{m}] \equiv 0$ on $\mathbb{S}_R$ for $R > 1$ or $R < 1$, respectively; such silent magnetizations are also frequently called annihilators).

If the magnetizations $\mathbf{m}, \overline{\mathbf{m}}$ are of the form (1.1), with susceptibilities $Q, \overline{Q}$ and dipole directions $\mathbf{d}, \overline{\mathbf{d}}$, then we say that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from inside/outside or we say that $(Q, \mathbf{d})$ is silent from inside/outside if the corresponding magnetizations $\mathbf{m}, \overline{\mathbf{m}}$ have these properties.

For us, the case $R > 1$ (i.e. equivalence/silence from outside) is of major relevance since we are eventually interested in using satellite magnetic field measurements, which are

obviously collected in the exterior of a planet. Now we can formulate the characterization of those contributions of $\mathbf{m}$ that are uniquely determined by knowledge of the potential $V[\mathbf{m}]$ using the notion of equivalent magnetizations.

Theorem 2.4: *Let $\mathbf{m} \in L^2(\mathbb{S}, \mathbb{R}^3)$ and its decomposition into $\tilde{\mathbf{m}}^{(1)}$, $\tilde{\mathbf{m}}^{(2)}$, $\tilde{\mathbf{m}}^{(3)}$ be given as in Theorem 2.2. Then, the following assertions hold true:*

(a) *The magnetization $\tilde{\mathbf{m}}^{(2)}$ is equivalent from outside to $\mathbf{m}$ while $\tilde{\mathbf{m}}^{(1)}$ is equivalent from inside to $\mathbf{m}$.*
(b) *The magnetization $\mathbf{m}$ is silent from outside if and only if $\tilde{\mathbf{m}}^{(2)} \equiv 0$ while $\mathbf{m}$ is silent from inside if and only if $\tilde{\mathbf{m}}^{(1)} \equiv 0$.*
(c) *If $\operatorname{supp}(\mathbf{m}) \subset \Gamma$, for a region $\Gamma \subset \mathbb{S}$ with $\overline{\Gamma} \neq \mathbb{S}$, then $\mathbf{m}$ is silent from outside if and only if it is silent from inside.*

We see that the contribution $\tilde{\mathbf{m}}^{(2)}$ is determined uniquely by $V = V[\mathbf{m}]$ on a sphere $\mathbb{S}_R$ of radius $R > 1$. If additionally $\operatorname{supp}(\mathbf{m}) \subset \Gamma$, then both $\tilde{\mathbf{m}}^{(1)}$ and $\tilde{\mathbf{m}}^{(2)}$ are determined uniquely. The latter follows from part (c) of the previous theorem: if $\mathbf{m}$ and $\overline{\mathbf{m}}$ were two magnetizations with $V[\mathbf{m}] = V[\overline{\mathbf{m}}]$ on $\mathbb{S}_R$, then $\mathbf{m} - \overline{\mathbf{m}}$ would be silent from outside with $\operatorname{supp}(\mathbf{m} - \overline{\mathbf{m}}) \subset \Gamma$. Thus, $\mathbf{m} - \overline{\mathbf{m}}$ must also be silent from inside and we get $\tilde{\mathbf{m}}^{(1)} = \tilde{\overline{\mathbf{m}}}^{(1)}$ and $\tilde{\mathbf{m}}^{(2)} = \tilde{\overline{\mathbf{m}}}^{(2)}$ from parts (a) and (b). Observing that $3(x \cdot \mathbf{d})x - \mathbf{d}|x|^2$ is non-tangential for all $x \in \mathbb{S}$ that are not orthogonal to $\mathbf{d}$, the next corollary is a direct consequence of Theorem 2.4 for dipole induced magnetizations.

Corollary 2.5: *Let $\mathbf{m} \in L^2(\mathbb{S}, \mathbb{R}^3)$ be of the induced form (1.1), with $Q \in L^2(\mathbb{S})$ and $\mathbf{d} \in \mathbb{S}$, and $\operatorname{supp}(Q) \subset \Gamma$ for a fixed region $\Gamma \subset \mathbb{S}$ with $\overline{\Gamma} \neq \mathbb{S}$. Then there does not exist another susceptibility $\overline{Q} \in L^2(\mathbb{S})$ with $\operatorname{supp}(\overline{Q}) \subset \Gamma$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \mathbf{d})$ are equivalent from outside or inside.*

In other words, a spatially localized susceptibility Q is uniquely determined by the knowledge of $V = V[Q, \mathbf{d}]$ on a sphere $\mathbb{S}_R$ of radius $R \neq 1$ if $\mathbf{d}$ is assumed to be given in advance. Next, we introduce two classical sets of vector spherical harmonics that reflect the decompositions from Theorems 2.1 and 2.2 in spectral domain. For details, the reader is referred to, e.g. [14–16,19].

Definition 2.6: For $n \in \mathbb{N}_0$, $k = -n, \ldots, n$, and $i = 1, 2, 3$, we set

$$\mathbf{y}_{n,k}^{(i)} = \left(\mu_n^{(i)}\right)^{-\frac{1}{2}} o^{(i)} Y_{n,k}$$

and

$$\tilde{\mathbf{y}}_{n,k}^{(i)} = \left(\tilde{\mu}_n^{(i)}\right)^{-\frac{1}{2}} \tilde{o}^{(i)} Y_{n,k},$$

with normalization constants $\mu_n^{(1)} = 1$, $\mu_n^{(2)} = \mu_n^{(3)} = n(n+1)$, and $\tilde{\mu}_n^{(1)} = (n+1)(2n+1)$, $\tilde{\mu}_n^{(2)} = n(2n+1)$, $\tilde{\mu}_n^{(3)} = n(n+1)$. The $Y_{n,k}$ denote an orthonormal set of scalar spherical harmonics (to be consistent with later computations in Section 4, we particularly choose $Y_{n,k}$ to be the complex-valued spherical harmonics as defined in [20,21]). It is to note that the type-(2) and type-(3) vector spherical harmonics vanish for degree $n = 0$ while this is not the case for type (1). To avoid introducing additional notation, the type-(2) and type-(3) vector spherical harmonics should, therefore, simply be regarded as void whenever they appear for degree $n = 0$.

The sets $\{\mathbf{y}_{n,k}^{(i)} : n \in \mathbb{N}_0, k = -n, \ldots, n, i = 1, 2, 3\}$ and $\{\tilde{\mathbf{y}}_{n,k}^{(i)} : n \in \mathbb{N}_0, k = -n, \ldots, n, i = 1, 2, 3\}$ each form a complete orthonormal system in $L^2(\mathbb{S}, \mathbb{R}^3)$. Thus, a Fourier expansion

$$\mathbf{m} = \sum_{i=1}^{3} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left(\tilde{m}^{(i)}\right)^{\wedge}(n,k) \tilde{\mathbf{y}}_{n,k}^{(i)}, \tag{2.13}$$

of a magnetization $\mathbf{m}$, with Fourier coefficients $(\tilde{m}^{(i)})^{\wedge}(n,k) = \int_{\mathbb{S}} \mathbf{m}(y) \cdot \tilde{\mathbf{y}}_{n,k}^{(i)}(y) \mathrm{d}\omega(y)$, inherits the properties of the Hardy-Hodge decomposition described in Theorem 2.4. For example., $\mathbf{m}$ is silent from outside if and only if all type-(2) Fourier coefficients vanish, i.e.

$$\left(\tilde{m}^{(2)}\right)^{\wedge}(n,k) = \int_{\mathbb{S}} \mathbf{m}(y) \cdot \tilde{\mathbf{y}}_{n,k}^{(2)}(y) \mathrm{d}\omega(y) = 0, \quad n \geq 1, k = -n, \ldots, n. \tag{2.14}$$

Analogously, $\mathbf{m}$ is silent from inside if and only if $(\tilde{m}^{(1)})^{\wedge}(n,k) = 0$ for all $n \geq 0, k = -n, \ldots, n$. Just as the Helmholtz and Hardy-Hodge decomposition in Theorem 2.2, the two sets of vector spherical harmonics have a simple connection: obviously $\tilde{\mathbf{y}}_{n,k}^{(3)} = \mathbf{y}_{n,k}^{(3)}$, and additionally

$$\tilde{\mathbf{y}}_{n,k}^{(1)} = \sqrt{\frac{n+1}{2n+1}} \mathbf{y}_{n,k}^{(1)} - \sqrt{\frac{n}{2n+1}} \mathbf{y}_{n,k}^{(2)}, \tag{2.15}$$

$$\tilde{\mathbf{y}}_{n,k}^{(2)} = \sqrt{\frac{n}{2n+1}} \mathbf{y}_{n,k}^{(1)} + \sqrt{\frac{n+1}{2n+1}} \mathbf{y}_{n,k}^{(2)}. \tag{2.16}$$

3. Spatially localized induced magnetizations

Let $Q, \overline{Q} \in L^2(\mathbb{S})$ with $\mathrm{supp}(Q), \mathrm{supp}(\overline{Q}) \subset \Gamma$ for a region $\Gamma \subset \mathbb{S}$ with closure $\overline{\Gamma} \neq \mathbb{S}$, and $\mathbf{d} \neq \pm\overline{\mathbf{d}} \in \mathbb{S}$. In order to check whether $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside, we are led to investigating if the residual magnetization

$$\mathbf{m}^{-}(x) = Q(\xi)\left(3(x \cdot \mathbf{d})x - \mathbf{d}\right) - \overline{Q}(x)\left(3(x \cdot \overline{\mathbf{d}})x - \overline{\mathbf{d}}\right), \quad x \in \mathbb{S}. \tag{3.1}$$

is silent from outside. According to Theorem 2.4, the latter would imply

$$\tilde{m}_1^{-} \equiv 0, \tag{3.2}$$

$$\tilde{m}_2^{-} \equiv 0, \tag{3.3}$$

which by Theorems 2.1 and 2.2 directly implies

$$m_1^{-} \equiv 0, \tag{3.4}$$

$$m_2^{-} \equiv 0, \tag{3.5}$$

where m_i^{-} and $\tilde{m}_i^{-}$, $i = 1, 2, 3$, denote the scalar functions appearing in the Helmholtz decomposition and the Hardy-Hodge decomposition of $\mathbf{m}^{-}$ according to Theorems 2.1

and 2.2, respectively. Equations (3.1) and (3.4) yield

$$m_1^-(x) = x \cdot \mathbf{m}^-(x) = 2(Q(x)\mathbf{d} \cdot x - \overline{Q}(x)\overline{\mathbf{d}} \cdot x) = 0, \quad x \in \mathbb{S}, \tag{3.6}$$

which can be reformulated to $\overline{Q}(x) = \frac{Q(x)}{x \cdot \overline{\mathbf{d}}} x \cdot \mathbf{d}$ and leads to the following representation of $\mathbf{m}^-$:

$$\mathbf{m}^-(x) = \frac{Q(x)}{x \cdot \overline{\mathbf{d}}} \left(\overline{\mathbf{d}}(x \cdot \mathbf{d}) - \mathbf{d}(x \cdot \overline{\mathbf{d}}) \right), \quad x \in \mathbb{S} \setminus \{y \in \mathbb{R}^3 : y \cdot \overline{\mathbf{d}} = 0\}. \tag{3.7}$$

For later reference, we define

$$P_{Q,\mathbf{d},\overline{\mathbf{d}}}(x) = \frac{Q(x)}{x \cdot \overline{\mathbf{d}}} x \cdot \mathbf{d}, \quad x \in \mathbb{S} \setminus \{y \in \mathbb{R}^3 : y \cdot \overline{\mathbf{d}} = 0\}. \tag{3.8}$$

Additionally, equations (3.4) and (3.5) imply that $\mathbf{m}^-$ has to be surface divergence-free if it is silent from outside, since it must hold $\mathbf{m}^- = \mathbf{m}_3^-$, where $\mathbf{m}_3^-$ is the vectorial surface divergence-free function of the Helmholtz and Hardy-Hodge decomposition of $\mathbf{m}^-$. Summarizing, we are led to the following assertion on uniqueness of dipole-induced magnetizations.

Lemma 3.1: *Let $Q \in L^2(\mathbb{S})$, with $\operatorname{supp}(Q) \subset \Gamma$, and $\mathbf{d} \in \mathbb{S}$. Then, for a given $\overline{\mathbf{d}} \neq \pm\mathbf{d} \in \mathbb{S}$, there exists a $\overline{Q} \in L^2(\mathbb{S})$ with $\operatorname{supp}(\overline{Q}) \subset \Gamma$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside if and only if $P_{Q,\mathbf{d},\overline{\mathbf{d}}} \in L^2(\mathbb{S})$ and $\mathbf{m}^-$ as in (3.7) is surface divergence-free.*

Remark 3.2: In particular, the lemma above implies that if $P_{Q,\mathbf{d},\overline{\mathbf{d}}} \notin L^2(\mathbb{S})$ and $\overline{\mathbf{d}} \neq \pm\mathbf{d}$, then there exists no other susceptibility $\overline{Q} \in L^2(\mathbb{S})$ with $\operatorname{supp}(\overline{Q}) \subset \Gamma$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside. This is a condition that should guarantee uniqueness for many geophysically relevant dipole induced magnetizations since non-uniqueness would require the susceptibility Q to be zero along at least one meridian.

Furthermore, in geophysical reality susceptibilities are non-negative. If Q is such a non-negative susceptibility, then, we know from the previous considerations that any equivalent susceptibility with $\operatorname{supp}(\overline{Q}) \subset \Gamma$ needs to be of the form $\overline{Q}(x) = \frac{Q(x)}{x \cdot \overline{\mathbf{d}}} x \cdot \mathbf{d}$. However, if $\overline{\mathbf{d}} \neq \mathbf{d}$, there always exist $x \in \mathbb{S}$ such that $x \cdot \mathbf{d}$ and $x \cdot \overline{\mathbf{d}}$ have different signs. If such an x where a sign change occurs is contained in $\operatorname{supp}(Q)$, then $\overline{Q}$ cannot be non-negative. In other words, the assumption of non-negative susceptibilities does supply some additional information but it does not generally lead to uniqueness of the problem.

In general, it is fairly easy to construct examples where non-uniqueness is given: Let $\mathbf{d} \neq \pm\overline{\mathbf{d}} \in \mathbb{S}$ and assume Q to be such that the function $P_{Q,\overline{\mathbf{d}}}$ given by $P_{Q,\overline{\mathbf{d}}}(x) = \frac{Q(x)}{x \cdot \overline{\mathbf{d}}}$ is continuously differentiable on $\mathbb{S}$. Then, in order for a $\overline{Q}$ with $\operatorname{supp}(\overline{Q}) \subset \Gamma$ to exist such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside, Lemma 3.1 implies that $\mathbf{m}^-$ as in (3.7) has to be surface divergence-free, i.e.

$$\left(\overline{\mathbf{d}}(x \cdot \mathbf{d}) - \mathbf{d}(x \cdot \overline{\mathbf{d}}) \right) \cdot \nabla^* P_{Q,\overline{\mathbf{d}}}(x) = \nabla^* \cdot \left(P_{Q,\overline{\mathbf{d}}}(x) \left(\overline{\mathbf{d}}(x \cdot \mathbf{d}) - \mathbf{d}(x \cdot \overline{\mathbf{d}}) \right) \right) = 0, \quad x \in \mathbb{S}. \tag{3.9}$$

A closer investigation of (3.9) shows that the spherical circles $\mathcal{C}_{t,\mathbf{d}\times\overline{\mathbf{d}}} = \{x \in \mathbb{S} : x \cdot \frac{\mathbf{d}\times\overline{\mathbf{d}}}{|\mathbf{d}\times\overline{\mathbf{d}}|} = t\}$, $t \in [-1, 1]$, represent the characteristic curves of the given differential equation and

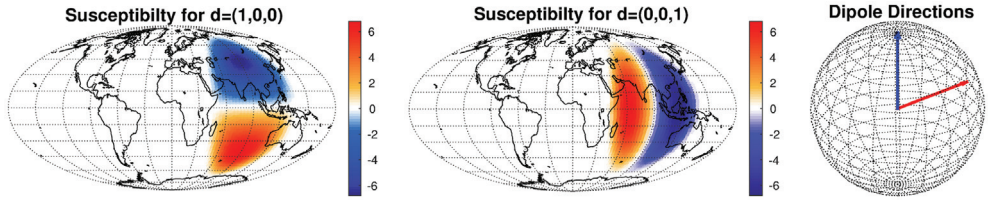

Figure 1. Illustration of two dipole induced magnetizations of the form described in Remark 3.2 that are equivalent from outside. We chose the auxiliary function to be $P(t) = e^{-\frac{1}{t^2}}\chi_{[0,1]}(t)$, the region $\Gamma = \{x \in \mathbb{S} : (0,-1,0)^T \cdot x \geq 0\}$ to be the eastern hemisphere, and the dipole directions $\mathbf{d} = (0,0,1)^T$ and $\overline{\mathbf{d}} = (1,0,0)^T$, respectively. *Left*: susceptibility Q, *Center*: susceptibility $\overline{Q}$, *Right*: dipole directions $\mathbf{d}$ (blue) and $\overline{\mathbf{d}}$ (red).

that $P_{Q,\overline{\mathbf{d}}}$ has to be constant along these curves. Thus, $P_{Q,\overline{\mathbf{d}}}$ has to be of the form $P_{Q,\overline{\mathbf{d}}}(x) = P\big(x \cdot \frac{\mathbf{d}\times\overline{\mathbf{d}}}{|\mathbf{d}\times\overline{\mathbf{d}}|}\big)$, where $P : [-1,1] \to \mathbb{R}$ is a continuously differentiable function with $P(t) = 0$ for all $t \in [-1,1]$ that satisfy $\mathcal{C}_{t,\mathbf{d}\times\overline{\mathbf{d}}} \cap (\mathbb{S} \setminus \Gamma) \neq \emptyset$. Given such a P, we see from Lemma 3.1 that

$$Q(x) = P\left(x \cdot \frac{\mathbf{d}\times\overline{\mathbf{d}}}{|\mathbf{d}\times\overline{\mathbf{d}}|}\right) x \cdot \overline{\mathbf{d}}, \quad x \in \mathbb{S},$$
$$\overline{Q}(x) = P\left(x \cdot \frac{\mathbf{d}\times\overline{\mathbf{d}}}{|\mathbf{d}\times\overline{\mathbf{d}}|}\right) x \cdot \mathbf{d}, \quad x \in \mathbb{S},$$

satisfy $\mathrm{supp}(Q), \mathrm{supp}(\overline{Q}) \subset \Gamma$ and that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside. An illustration of two such magnetizations, with $\Gamma \subset \mathbb{S}$ being the eastern hemisphere, is shown in Figure 1.

Let now $\overline{\mathbf{m}} \in H_1(\mathbb{S}, \mathbb{R}^3)$, with $\mathrm{supp}(\overline{\mathbf{m}}) \subset \Gamma$, and $V = V[\overline{\mathbf{m}}]$ be the corresponding potential on $\mathbb{S}_R$. We are interested in finding out if there exists a dipole induced magnetization of the form (1.1) that produces the same magnetic potential on $\mathbb{S}_R$ as $\overline{\mathbf{m}}$ (which is not necessarily of dipole induced form). If there exist $Q \in H_1(\mathbb{S})$, with $\mathrm{supp}(Q) \subset \Gamma$, and $\mathbf{d} \in \mathbb{S}$ such that the corresponding magnetization $\mathbf{m}(x) = Q(x)(3(x \cdot \mathbf{d})x - \mathbf{d})$ is equivalent from outside to $\overline{\mathbf{m}}$ (we also say $(Q, \mathbf{d})$ is equivalent from outside to $\overline{\mathbf{m}}$), then Theorem 2.4 together with the Helmholtz and the Hardy-Hodge decomposition tells us

$$x \cdot \overline{\mathbf{m}}(x) = \overline{m}_1(x) = m_1(x) = 2Q(x)x \cdot \mathbf{d}, \quad x \in \mathbb{S}, \tag{3.10}$$

where $\overline{m}_1$ and m_1 are the radial contributions of $\overline{\mathbf{m}}$ and $\mathbf{m}$, respectively. The higher smoothness assumption of $\overline{\mathbf{m}}, \mathbf{m} \in H_1(\mathbb{S}, \mathbb{R}^3)$ is only required to allow differentiation of $\overline{\mathbf{m}}$ and $\mathbf{m}$ later on. From (3.10) we get for the susceptibility Q that

$$Q(x) = \frac{x \cdot \overline{\mathbf{m}}(x)}{2x \cdot \mathbf{d}}, \quad x \in \mathbb{S} \setminus \{y \in \mathbb{R}^3 : y \cdot \mathbf{d} = 0\}. \tag{3.11}$$

It remains to find the dipole direction $\mathbf{d} \in \mathbb{S}$. Again, referring to Theorem 2.4 and the decompositions from Theorems 2.1 and 2.2, we get, additionally to $\overline{\mathbf{m}}_1 = \mathbf{m}_1$, that $\overline{\mathbf{m}}_2 = \nabla^* \overline{m}_2 = \nabla^* m_2 = \mathbf{m}_2$. This yields, together with (3.11),

$$\begin{aligned}\Delta^*\overline{m}_2(x) &= \nabla^* \cdot \big(\overline{\mathbf{m}}(x) - \overline{\mathbf{m}}_1(x)\big) = \nabla^* \cdot \big(Q(x)(3(x\cdot\mathbf{d})x - \mathbf{d}) - \mathbf{m}_1(x)\big)\\ &= \nabla^* \cdot \left(\frac{3}{2}x\,\overline{m}_1(x) - \frac{\overline{m}_1(x)}{2(x\cdot\mathbf{d})}\mathbf{d} - x\,\overline{m}_1(x)\right)\\ &= \overline{m}_1(x) - \frac{d}{2(x\cdot\mathbf{d})}\cdot\nabla^*\overline{m}_1(x) + \frac{\mathbf{d}}{2(x\cdot\mathbf{d})^2}\cdot(\mathbf{d} - (x\cdot\mathbf{d})x)\overline{m}_1(x)\\ &= \frac{\overline{m}_1(x)}{2} - \frac{\mathbf{d}}{2(x\cdot\mathbf{d})}\cdot\nabla^*\overline{m}_1(x) + \frac{\overline{m}_1(x)}{2(x\cdot\mathbf{d})^2}, \quad x\in\mathbb{S}\setminus\{y\in\mathbb{R}^3 : y\cdot\mathbf{d}=0\}.\end{aligned}$$

Multiplying the above by $2(x\cdot\mathbf{d})^2$ leads to the condition

$$2(x\cdot\mathbf{d})^2\Delta^*\overline{m}_2(x) - (x\cdot\mathbf{d})^2\overline{m}_1(x) + (x\cdot\mathbf{d})\mathbf{d}\cdot\nabla^*\overline{m}_1(x) - \overline{m}_1(x) = 0. \tag{3.12}$$

Eventually, integrating the square of the left hand side of (3.12) over $\mathbb{S}$, we find that the dipole direction $\mathbf{d}\in\mathbb{S}$ has to be a zero of the following fourth-order polynomial

$$\begin{aligned}T_{\overline{g},\overline{\mathbf{h}},\overline{m}_1}(\mathbf{d}) = &\sum_{i,j,k,l=1}^{3} d_id_jd_kd_l\int_{\mathbb{S}} y_iy_jy_ky_l|\overline{g}(y)|^2\mathrm{d}\omega(y) + \sum_{i,j,k,l=1}^{3} d_id_jd_kd_l\int_{\mathbb{S}} y_iy_j\,\overline{h}_k(y)\overline{h}_l(y)\mathrm{d}\omega(y)\\ &+2\sum_{i,j,k,l=1}^{3} d_id_jd_kd_l\int_{\mathbb{S}} y_iy_jy_k\,\overline{h}_l(y)\overline{g}(y)\mathrm{d}\omega(y) - 2\sum_{i,j=1}^{3} d_id_j\int_{\mathbb{S}} y_iy_j\,\overline{m}_1(y)\overline{g}(y)\mathrm{d}\omega(y)\\ &-2\sum_{i,j=1}^{3} d_id_j\int_{\mathbb{S}} y_i\,\overline{h}_j(y)\overline{m}_1(y)\mathrm{d}\omega(y) + \int_{\mathbb{S}}|\overline{m}_1(y)|^2\mathrm{d}\omega(y),\end{aligned} \tag{3.13}$$

where

$$\overline{\mathbf{h}} = (\overline{h}_1,\overline{h}_2,\overline{h}_3)^T = \nabla^*\overline{m}_1, \tag{3.14}$$

$$\overline{g} = 2\Delta^*\overline{m}_2 - \overline{m}_1, \tag{3.15}$$

and $\mathbf{d} = (d_1,d_2,d_3)^T, y = (y_1,y_2,y_3)^T$. We can now summarize these observations in the following theorem.

Theorem 3.3: *Let $\overline{\mathbf{m}}\in H_1(\mathbb{S},\mathbb{R}^3)$ with $\mathrm{supp}(\overline{\mathbf{m}})\subset\Gamma$, for some region $\Gamma\subset\mathbb{S}$ with $\overline{\Gamma}\neq\mathbb{S}$. Furthermore, we set $P_{\overline{\mathbf{m}},\mathbf{d}}(x) = \frac{x\cdot\overline{\mathbf{m}}(x)}{2x\cdot\mathbf{d}}$. Then there exists a susceptibility $Q\in H_1(\mathbb{S})$ with $\mathrm{supp}(Q)\subset\Gamma$ and a dipole direction $\mathbf{d}\in\mathbb{S}$ such that $(Q,\mathbf{d})$ is equivalent to $\overline{\mathbf{m}}$ from outside if and only if there exists a $\mathbf{d}\in\mathbb{S}$ that satisfies*

$$T_{\overline{g},\overline{\mathbf{h}},\overline{m}_1}(\mathbf{d}) = 0$$

and $P_{\overline{\mathbf{m}},\mathbf{d}}\in H_1(\mathbb{S})$, where $T_{\overline{g},\overline{\mathbf{h}},\overline{m}_1}$ is given as in (3.13).

Remark 3.4: On the one hand, Theorem 3.3 provides a means of deciding whether a given potential V on $\mathbb{S}_R$ can be produced by a dipole induced magnetization of the form (1.1). Namely, one first inverts V to find a general magnetization $\overline{\mathbf{m}}$ such that $V = V[\overline{\mathbf{m}}]$ on $\mathbb{S}_R$. Afterwards one can use Theorem 3.3 to check whether V can also be expressed in the form $V[Q,\mathbf{d}]$. On the other hand, Theorem 3.3 can give hints at the uniqueness of the susceptibility Q and dipole direction $\mathbf{d}$: if $T_{\overline{g},\overline{\mathbf{h}},\overline{m}_1}$ has only one zero (up to the sign), then uniqueness is given.

4. Band-limited induced magnetizations

Analogous questions as in Section 3 are investigated under the assumption that the magnetization $\mathbf{m}$ is band-limited (and not spatially localized in the sense $\mathrm{supp}(\mathbf{m})\subset\Gamma$).

Definition 4.1: We call a function $\mathbf{f} \in L^2(\mathbb{S}, \mathbb{R}^3)$ *band-limited* if there exists a $N \in \mathbb{N}_0$ such that

$$\left(\tilde{f}^{(i)}\right)^\wedge(n,k) = \int_{\mathbb{S}} \mathbf{f}(y) \cdot \tilde{\mathbf{y}}^{(i)}_{n,k}(y) \mathrm{d}\omega(y) = 0, \quad n \geq N+1, k = -n, \ldots, n, \; i = 1,2,3.,$$

i.e. all Fourier coefficients vanish from some degree $N+1$ on. N is called the band-limit of $\mathbf{f}$. A scalar function $f \in L^2(\mathbb{S})$ is called *band-limited* if there exists a $N \in \mathbb{N}_0$ such that

$$f^\wedge(n,k) = \int_{\mathbb{S}} f(y) Y_{n,k}(y) \mathrm{d}\omega(y) = 0, \quad n \geq N+1, k = -n, \ldots, n.$$

We start by computing the Fourier expansion of magnetizations $\mathbf{m}$ of the form (1.1). The inducing vectorial dipole field part can be expressed as

$$3(\mathbf{d} \cdot x)x - \mathbf{d} = \sum_{k=-1}^{1} \frac{8\pi}{3} Y_{1,k}(\mathbf{d}) \left(\mathbf{y}^{(1)}_{1,k}(x) - \frac{1}{\sqrt{2}} \mathbf{y}^{(2)}_{1,k}(x) \right), \quad x \in \mathbb{S}. \tag{4.1}$$

For a susceptibility $Q \in L^2(\mathbb{S})$ with Fourier expansion $Q = \sum_{m=0}^{\infty} \sum_{l=-m}^{m} Q^\wedge(m,l) Y_{m,l}$ one can then use the calculus of Wigner symbols (e.g. [19,22]; a notation compatible with ours is used in [20]) to obtain the following expression for the corresponding dipole induced magnetization $\mathbf{m}(x) = Q(x)\left(3(\mathbf{d} \cdot x)x - \mathbf{d}\right)$:

$$\mathbf{m}(x) = \frac{8\pi}{3} \sum_{i=1}^{3} \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{m=p-1}^{p+1} \sum_{l=q-1}^{q+1} Q^\wedge(m,l) Y_{1,k_{l,q}}(\mathbf{d}) \alpha^{(i)}_{p,q,m,l} \mathbf{y}^{(i)}_{pq}(x), \quad x \in \mathbb{S}, \tag{4.2}$$

where $k_{l,q} = -1$ if $l = q+1$, $k_{l,q} = 0$ if $l = q$, and $k_{l,q} = 1$ if $l = q-1$, and

$$\alpha^{(1)}_{p,q,m,l} = (-1)^q \sqrt{\frac{3(2m+1)(2p+1)}{4\pi}} \begin{pmatrix} 1 & p & m \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & p & m \\ k_{l,q} & -q & l \end{pmatrix} \tag{4.3}$$

$$\alpha^{(2)}_{p,q,m,l} = -\frac{1}{4\sqrt{p(p+1)}} (2 - m(m+1) + p(p+1)) \alpha^{(1)}_{p,q,m,l} \tag{4.4}$$

$$\begin{aligned} \alpha^{(3)}_{p,q,m,l} = {} & i(-1)^{k_{l,q}} (2p+1) \sqrt{\frac{2m+1}{4\pi}} \begin{pmatrix} p & m & 1 \\ -q & l & k_{l,q} \end{pmatrix} \\ & \times \left(\begin{Bmatrix} p & p & 1 \\ 0 & 1 & m \end{Bmatrix} \begin{pmatrix} p & m & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sqrt{\frac{5}{2}} \begin{Bmatrix} p & p & 1 \\ 2 & 1 & m \end{Bmatrix} \begin{pmatrix} p & m & 2 \\ 0 & 0 & 0 \end{pmatrix} \right). \end{aligned} \tag{4.5}$$

The brackets $\left\{ \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right\}$ denote Wigner-6j symbols while $\left(\begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right)$ denote Wigner-3j symbols. It is to note that round brackets are also used for matrices, however, it should be clear from the context if we mean Wigner-3j symbols or matrices.

An expansion of the magnetization $\mathbf{m}$ in terms of $\tilde{\mathbf{y}}^{(i)}_{n,k}$, which reflects the Hardy-Hodge decomposition from Theorem 2.2, can be directly obtained from (4.2), (2.15), and (2.16). This is summarized in the following proposition.

Proposition 4.2: *Let $Q \in L^2(\mathbb{S})$, $\mathbf{d} \in \mathbb{S}$, and $\alpha^{(i)}_{p,q,m,l}$, $i = 1,2,3$, be the coefficients as in (4.3)–(4.5). Then, the dipole induced magnetization $\mathbf{m}(x) = Q(x)\left(3(\mathbf{d} \cdot x)x - \mathbf{d}\right)$ has the Fourier expansion*

$$\mathbf{m}(x) = \frac{8\pi}{3} \sum_{i=1}^{3} \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \sum_{m=p-1}^{p+1} \sum_{l=q-1}^{q+1} Q^\wedge(m,l) Y_{1,k_{l,q}}(\mathbf{d}) \tilde{\alpha}^{(i)}_{p,q,m,l} \tilde{\mathbf{y}}^{(i)}_{pq}(x), \quad x \in \mathbb{S},$$

with

$$\begin{aligned} \tilde{\alpha}^{(1)}_{p,q,m,l} &= \sqrt{\frac{p+1}{2p+1}} \alpha^{(1)}_{p,q,m,l} - \sqrt{\frac{p}{2p+1}} \alpha^{(2)}_{p,q,m,l} \\ \tilde{\alpha}^{(2)}_{p,q,m,l} &= \sqrt{\frac{p}{2p+1}} \alpha^{(1)}_{p,q,m,l} + \sqrt{\frac{p+1}{2p+1}} \alpha^{(2)}_{p,q,m,l} \\ \tilde{\alpha}^{(3)}_{p,q,m,l} &= \alpha^{(3)}_{p,q,m,l}. \end{aligned}$$

The properties of the Wigner-3j symbols yield that $\tilde{\alpha}^{(1)}_{p,q,m,l} = \tilde{\alpha}^{(2)}_{p,q,m,l} = 0$ *if* $p = m$*, so that the fourth sum in the above representation of* $\mathbf{m}$ *has contributions only for* $m \in \{p-1, p+1\}$*. Any Fourier coefficients* $Q^\wedge(m,l) = \int_{\mathbb{S}} Q(y) Y_{m,l}(y) \mathrm{d}\omega(y)$ *with* $l \geq m+1$ *or* $l \leq -m-1$ *are zero by definition.*

Now we are in a place to characterize silent band-limited dipole induced magnetizations. Theorem 2.4(b) essentially states that a magnetization $\mathbf{m}$ is silent from outside if and only if all type-(2) Fourier coefficients $(\tilde{m}^{(2)})^\wedge(p,q)$ vanish. In consequence, the representation in Proposition 4.2 implies that $\mathbf{m}$ is silent from outside if and only if

$$\sum_{l=q-1}^{q+1} Q^\wedge(p-1,l) Y_{1,k_{l,q}}(\mathbf{d}) \tilde{\alpha}^{(2)}_{p,q,p-1,l} + \sum_{l=q-1}^{q+1} Q^\wedge(p+1,l) Y_{1,k_{l,q}}(\mathbf{d}) \tilde{\alpha}^{(2)}_{p,q,p+1,l} = 0, \qquad p \geq 1, q = -p, \ldots, p. \tag{4.6}$$

For $p \geq 1, q = -p, \ldots, p$, and $l \in \{q-1, q, q+1\}$, we can compute from the representation in Proposition 4.2 that

$$\tilde{\alpha}^{(2)}_{p,q,p-1,l} = \frac{(-1)^q}{2}(p-1)\sqrt{\frac{3(2p-1)}{4\pi p}} \begin{pmatrix} 1 & p & p-1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & p & p-1 \\ k_{l,q} & -q & l \end{pmatrix}, \tag{4.7}$$

so that $\tilde{\alpha}^{(2)}_{p,q,p-1,l} = 0$ if and only if $p = 1$. Analogously, one can see that $\tilde{\alpha}^{(2)}_{p,q,p+1,l} \neq 0$ for all $p \geq 1$, $q = -p, \ldots, p$, and $l \in \{q-1, q, q+1\}$. This leads us to the following statement.

Lemma 4.3: *Let* $Q \in L^2(\mathbb{S})$ *be band-limited and* $\mathbf{d} \in \mathbb{S}$*. If* $\overline{Q} \in L^2(\mathbb{S})$ *is another band-limited susceptibility such that* $(Q, \mathbf{d})$ *and* $(\overline{Q}, \mathbf{d})$ *are equivalent from outside, then all Fourier coefficients for degrees greater or equal to one coincide, i.e.* $Q^\wedge(m,l) = \overline{Q}^\wedge(m,l)$ *for all* $m \geq 1$, $l = -m, \ldots, m$.

***Proof*:** Let us assume for now that $(Q, \mathbf{d})$ is silent from outside. The equations in (4.6) can be rewritten in the form

$$\mathbf{M}^{\mathbf{y_d}}_{p-1} \mathbf{q}_{p-1} + \mathbf{N}^{\mathbf{y_d}}_{p+1} \mathbf{q}_{p+1} = 0, \quad p \geq 1, \tag{4.8}$$

where $\mathbf{M}^{\mathbf{y_d}}_{p-1} \in \mathbb{C}^{(2p+1)\times(2p-1)}$, $\mathbf{N}^{\mathbf{y_d}}_{p+1} \in \mathbb{C}^{(2p+1)\times(2p+3)}$ are tri-band matrices and $\mathbf{q}_{p-1} \in \mathbb{C}^{2p-1}$, $\mathbf{q}_{p+1} \in \mathbb{C}^{2p+3}$ vectors. More precisely, the matrix $\mathbf{M}^{\mathbf{y_d}}_{p-1}$ and the vector $\mathbf{q}_{p-1}$ are of the form

$$\mathbf{M}^{\mathbf{y_d}}_{p-1} = \begin{pmatrix} a_{-p} & 0 & \cdots & 0 \\ b_{-p+1} & a_{-p+1} & \ddots & \vdots \\ c_{-p+2} & b_{-p+2} & \ddots & 0 \\ 0 & \ddots & \ddots & a_{p-2} \\ \vdots & \ddots & \ddots & b_{p-1} \\ 0 & \cdots & 0 & c_p \end{pmatrix}, \quad \mathbf{q}_{p-1} = \begin{pmatrix} Q^\wedge(p-1, -(p-1)) \\ \vdots \\ Q^\wedge(p-1, p-1) \end{pmatrix}, \tag{4.9}$$

with $a_q = \tilde{\alpha}^{(2)}_{p,q,p-1,q+1} y_{\mathbf{d},1}$, $b_q = \tilde{\alpha}^{(2)}_{p,q,p-1,q} y_{\mathbf{d},2}$, $c_q = \tilde{\alpha}^{(2)}_{p,q,p-1,q-1} y_{\mathbf{d},3}$, and the auxiliary vector $\mathbf{y_d} = (y_{\mathbf{d},1}, y_{\mathbf{d},2}, y_{\mathbf{d},3}) = (Y_{1,-1}(\mathbf{d}), Y_{1,0}(\mathbf{d}), Y_{1,1}(\mathbf{d}))$. The matrix $\mathbf{N}^{\mathbf{y_d}}_{p+1}$ has the form

$$\mathbf{N}^{\mathbf{y_d}}_{p+1} = \begin{pmatrix} \gamma_{-p} & \beta_{-p} & \alpha_{-p} & 0 & \cdots & 0 \\ 0 & \gamma_{-p+1} & \beta_{-p+1} & \alpha_{-p+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_p & \beta_p & \alpha_p \end{pmatrix}, \quad \mathbf{q}_{p+1} = \begin{pmatrix} Q^\wedge(p+1, -(p+1)) \\ \vdots \\ Q^\wedge(p+1, p+1) \end{pmatrix}, \tag{4.10}$$

with $\alpha_q = \tilde{\alpha}^{(2)}_{p,q,p+1,q+1} y_{\mathbf{d},1}$, $\beta_q = \tilde{\alpha}^{(2)}_{p,q,p+1,q} y_{\mathbf{d},2}$, $\gamma_q = \tilde{\alpha}^{(2)}_{p,q,p+1,q-1} y_{\mathbf{d},3}$. For $p \geq 1$ we have seen in (4.7) that $\tilde{\alpha}^{(2)}_{p,q,p+1,l} \neq 0$ and for $p \geq 2$ that $\tilde{\alpha}^{(2)}_{p,q,p-1,l} \neq 0$. Furthermore, for any $\mathbf{d} \in \mathbb{S}$, at least one

of the expressions $Y_{1,-1}(\mathbf{d})$, $Y_{1,0}(\mathbf{d})$, $Y_{1,1}(\mathbf{d})$ is non-zero. In consequence, for $p \geq 2$, the entries of at least one of the three main diagonals of $\mathbf{M}_{p-1}^{\mathbf{y_d}}$ are all non-zero, so that the matrix has full rank, i.e. $\operatorname{rank}(\mathbf{M}_{p-1}^{\mathbf{y_d}}) = 2p-1$. The same holds true for $\mathbf{N}_{p+1}^{\mathbf{y_d}}$.

Since Q is band-limited, there must exist a $N \in \mathbb{N}_0$ such that $\mathbf{q}_{p+1} = 0$, for $p \geq N$. Thus, iteratively, we obtain from (4.8) and the full rank of $\mathbf{M}_{p-1}^{\mathbf{y_d}}$ that any band-limited dipole induced magnetization that is silent from outside has to satisfy $\mathbf{q}_{p-1} = 0$, for $p \geq 2$, i.e. $Q^\wedge(m,l) = 0$ for all $m \geq 1$, $l = -m, \ldots, m$.

If $(Q, \mathbf{d})$ is not silent from outside but Q and $\overline{Q}$ are two susceptibilities such that $(Q, \mathbf{d})$ and $(\overline{Q}, \mathbf{d})$ are equivalent from outside, then the difference of the two corresponding magnetizations must be silent from outside, i.e. it must be satisfied that

$$\mathbf{M}_{p-1}^{\mathbf{y_d}}(\mathbf{q}_{p-1} - \overline{\mathbf{q}}_{p-1}) + \mathbf{N}_{p+1}^{\mathbf{y_d}}(\mathbf{q}_{p+1} - \overline{\mathbf{q}}_{p+1}) = 0, \tag{4.11}$$

for all $p \geq 1$. Now, the previous considerations imply the statement of the lemma. □

Remark 4.4: Equation (4.6) contains contributions of the Fourier coefficient $Q^\wedge(0,0)$ only for the choice $p = 1$. The observations in (4.7), however, yield that $\tilde{\alpha}_{1,q,0,0}^{(2)} = 0$, $q \in \{-1, 0, 1\}$, so that $Q^\wedge(0,0)$ does not have any effect on the magnetic potential $V[Q, \mathbf{d}]$ on $\mathbb{S}_R$, $R > 1$. In other words, any constant susceptibility Q leads to a dipole induced magnetization that is silent from outside. Lemma 4.3 implies that those are all silent band-limited dipole induced magnetizations.

If the particular dipole direction $\mathbf{d} = (0,0,1)$ is chosen, then $Y_{1,k_{l,q}}(\mathbf{d}) = 0$ for $k_{l,q} \neq 0$. For this setting, the equations (4.6) reduce to

$$Q^\wedge(p-1,q)Y_{1,0}(\mathbf{d})\tilde{\alpha}_{p,q,p-1,q}^{(2)} + Q^\wedge(p+1,q)Y_{1,0}(\mathbf{d})\tilde{\alpha}_{p,q,p+1,q}^{(2)} = 0, \quad p \in \mathbb{N}_0, q = -p, \ldots, p. \tag{4.12}$$

Latter is essentially identical to the recursion relation that was obtained in [1] to characterize silent magnetizations (which they called annihilators). In this sense, Lemma 4.3 and the first part of this remark are just slightly more general statements of these results.

Next, we are interested in the equivalence of two dipole induced magnetizations with possibly different dipole directions. More precisely, for a given band-limited $Q \in L^2(\mathbb{S})$ and $\mathbf{d} \in \mathbb{S}$, we want to determine if there exists another susceptibility $\overline{Q} \in L^2(\mathbb{S})$ and dipole direction $\overline{\mathbf{d}} \neq \pm\mathbf{d} \in \mathbb{S}$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside (for $\overline{\mathbf{d}} = \pm\mathbf{d}$ this is, of course, always possible by Lemma 4.3 and Remark 4.4). Equivalence from outside means that the residual magnetization

$$\mathbf{m}^-(x) = Q(\xi)\left(3(x \cdot \mathbf{d})x - \mathbf{d}\right) - \overline{Q}(x)\left(3(x \cdot \overline{\mathbf{d}})x - \overline{\mathbf{d}}\right), \quad x \in \mathbb{S}. \tag{4.13}$$

is silent from outside. According to (4.6) and (4.8) this is possible if and only if

$$\mathbf{M}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} + \mathbf{N}_{p+1}^{\mathbf{y_d}}\mathbf{q}_{p+1} - \mathbf{M}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\overline{\mathbf{q}}_{p-1} - \mathbf{N}_{p+1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\overline{\mathbf{q}}_{p+1} = 0, \quad p \geq 1. \tag{4.14}$$

The quantities $\mathbf{M}_{p-1}^{\mathbf{y_d}}$, $\mathbf{N}_{p+1}^{\mathbf{y_d}}$, $\mathbf{q}_{p\pm1}$ are defined as in (4.9) and (4.10). $\overline{\mathbf{q}}_{p\pm1}$ denotes the counterpart of $\mathbf{q}_{p\pm1}$ corresponding to $\overline{Q}$. Since the susceptibilities Q, $\overline{Q}$ are assumed to be band-limited, there exists some $N \in \mathbb{N}_0$ such that $\mathbf{q}_{p+1} = \overline{\mathbf{q}}_{p+1} = 0$ for all $p \geq N$, so that, for $p \in \{N, N+1\}$, (4.14) reduces to

$$\mathbf{M}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} - \mathbf{M}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\overline{\mathbf{q}}_{p-1} = 0. \tag{4.15}$$

Now, given $\mathbf{q}_{p-1}$ and $\mathbf{d}$, the first question to answer is if there exist $\overline{\mathbf{q}}_{p-1} \in \mathbb{C}^{2p-1}$ and a dipole direction $\overline{\mathbf{d}} \neq \pm\mathbf{d} \in \mathbb{S}$ such that (4.15) is satisfied. The system of linear equations is overdetermined, but from the proof of Lemma 4.3 we know that $\mathbf{M}_{p-1}^{\mathbf{y_d}}$ has full rank. From now on, we assume that $\overline{\mathbf{d}} \in \mathbb{S} \setminus \{(0,0,\pm1)\}$ because then $Y_{1,1}(\overline{\mathbf{d}}) \neq 0$. This yields that the matrix $\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}$, which is obtained from $\mathbf{M}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}$ by deleting the first two rows, is invertible. Analogously, $\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}$ denotes $\mathbf{M}_{p-1}^{\mathbf{y_d}}$ with its first two rows deleted. The uniquely determined candidate for $\overline{\mathbf{q}}_{p-1} \in \mathbb{C}^{2p-1}$, $p \in \{N, N+1\}$, is then obtained by

$$\overline{\mathbf{q}}_{p-1} = \left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\right)^{-1}\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1}. \tag{4.16}$$

It remains to check whether (4.15) is valid for this $\overline{\mathbf{q}}_{p-1}$, i.e. if

$$\mathbf{M}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} - \mathbf{M}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\right)^{-1}\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} = 0 \tag{4.17}$$

holds true for $p \in \{N, N+1\}$. By construction it actually suffices to check if the first two rows of the above system of equations hold true. From the structure of the inverse of upper triangular matrices, we find that $(\Pi_{q=-p+2}^{p} c_q(p, \mathbf{y}_{\overline{\mathbf{d}}}))(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}})^{-1}$ is a matrix with entries that are polynomials with respect to $\mathbf{y}_{\overline{\mathbf{d}}} = (Y_{1,-1}(\overline{\mathbf{d}}), Y_{1,0}(\overline{\mathbf{d}}), Y_{1,1}(\overline{\mathbf{d}})) \in \mathbb{C}^3$ (the coefficients $c_q = c_q(p, \mathbf{y}_{\overline{\mathbf{d}}})$ are defined as in (4.9) and depend on p and $\mathbf{y}_{\overline{\mathbf{d}}}$). In conclusion, if $\overline{\mathbf{d}} \in \mathbb{S} \setminus \{(0,0,\pm 1)\}$ is an admissible candidate for a dipole direction, then $\mathbf{y}_{\overline{\mathbf{d}}}$ has to be a zero of the vector-valued polynomials $\mathbf{R}_{p,Q,\mathbf{d}}$ given by

$$\mathbf{R}_{p,Q,\mathbf{d}}(\mathbf{y}) = (\Pi_{q=-p+2}^{p} c_q(p, \mathbf{y}))\left(\mathbf{M}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} - \mathbf{M}_{p-1}^{\mathbf{y}}\left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}}\right)^{-1}\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1}\right), \quad \mathbf{y} \in \mathbb{C}^3, \tag{4.18}$$

for $p \in \{N, N+1\}$. It remains to check the cases $p = 2, \ldots, N-1$ ($p = 1$ is not of interest since the coefficients $Q^{\wedge}(0,0)$ and $\overline{Q}^{\wedge}(0,0)$ can be chosen arbitrarily according to Lemma 4.3 and Remark 4.4). In this case, the second and fourth summand in (4.14) cannot be omitted and we get that $\mathbf{y}_{\overline{\mathbf{d}}}$ additionally needs to be a zero of the polynomials $\mathbf{S}_{p,Q,\mathbf{d}}$ given by

$$\begin{aligned}\mathbf{S}_{p,Q,\mathbf{d}}(\mathbf{y}) = (\Pi_{n=p}^{N}\Pi_{q=-n+2}^{n} c_q(n, \mathbf{y}))\Big(&\mathbf{M}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} + \mathbf{N}_{p+1}^{\mathbf{y_d}}\mathbf{q}_{p+1} - \mathbf{N}_{p+1}^{\mathbf{y}}\overline{\mathbf{q}}_{p+1} \\ &- \mathbf{M}_{p-1}^{\mathbf{y}}\left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}}\right)^{-1}\left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} + \widehat{\mathbf{N}}_{p+1}^{\mathbf{d}}\mathbf{q}_{p+1} - \widehat{\mathbf{N}}_{p+1}^{\mathbf{y}}\overline{\mathbf{q}}_{p+1}\right)\Big),\end{aligned} \tag{4.19}$$

for $p = 2, \ldots, N-1$. The additional product $\Pi_{n=p}^{N}$ is only included to guarantee that $\mathbf{S}_{p,Q,\mathbf{d}}$ is a polynomial, although this is not crucial for our statements. The required vectors $\overline{\mathbf{q}}_{p+1}$ can be computed iteratively from the results of the previous steps: starting with (4.16) for $p \in \{N, N+1\}$ and continuing with

$$\overline{\mathbf{q}}_{p-1} = \left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\right)^{-1}\left(\widehat{\mathbf{M}}_{p-1}^{\mathbf{y_d}}\mathbf{q}_{p-1} + \widehat{\mathbf{N}}_{p+1}^{\mathbf{d}}\mathbf{q}_{p+1} - \widehat{\mathbf{N}}_{p+1}^{\mathbf{y}_{\overline{\mathbf{d}}}}\overline{\mathbf{q}}_{p+1}\right), \tag{4.20}$$

for $p = N-1, \ldots, 2$.

Eventually, we see that in order to determine if, for a given band-limited susceptibility Q and dipole direction $\mathbf{d} \in \mathbb{S}$, there exists another band-limited susceptibility $\overline{Q}$ and dipole direction $\overline{\mathbf{d}} \in \mathbb{S}\setminus\{(0,0,\pm 1)\}$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside, one possible way is to find common zeros of $\mathbf{R}_{p,Q,\mathbf{d}}$, $p \in \{N, N+1\}$, and $\mathbf{S}_{p,Q,\mathbf{d}}$, $p = 2, \ldots, N-1$. If a common zero $\mathbf{y} \in \mathbb{C}^3$ other than $\mathbf{y} = \mathbf{y}_{\pm\mathbf{d}}$ exists and if it is of the form $\mathbf{y} = \mathbf{y}_{\overline{\mathbf{d}}} = (Y_{1,-1}(\overline{\mathbf{d}}), Y_{1,0}(\overline{\mathbf{d}}), Y_{1,1}(\overline{\mathbf{d}}))$, then a candidate for $\overline{\mathbf{d}}$ has been found (and the corresponding suceptibility $\overline{Q}$ is determined up to a constant via the Fourier coefficients gathered in (4.16), (4.20)). However, it is by no means true that all common zeros of $\mathbf{R}_{p,Q,\mathbf{d}}$ and $\mathbf{S}_{p,Q,\mathbf{d}}$ need to be representable in the form $(Y_{1,-1}(\overline{\mathbf{d}}), Y_{1,0}(\overline{\mathbf{d}}), Y_{1,1}(\overline{\mathbf{d}}))$ in the first place. Finally, the so far excluded case $\overline{\mathbf{d}} = (0,0,\pm 1)$ has to be checked separately (e.g. by choosing $\widehat{\mathbf{M}}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}$ to be the matrix that is obtained from $\mathbf{M}_{p-1}^{\mathbf{y}_{\overline{\mathbf{d}}}}$ not by deleting the first two rows but by deleting the first and last row).

Remark 4.5: From Lemma 4.3 and Remark 4.4 it is clear that for constant susceptibilities $Q \equiv c$ and $\overline{Q} \equiv \overline{c}$, with $c, \overline{c} \neq 0$, it holds that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside for any $\mathbf{d}, \overline{\mathbf{d}} \in \mathbb{S}$. A slightly more complex example for equivalent band-limited magnetizations would be for band-limit $N = 1$. Let us choose $\mathbf{d} = (0,0,1)$ and $\overline{\mathbf{d}} = (1,0,0)$ and construct $\mathbf{q}_1$ and $\overline{\mathbf{q}}_1$ from (4.16) and (4.17). Clearly, $\mathbf{q}_1$ needs to be in the nullspace of of the matrix $\mathbf{M}_1^{\mathbf{y_d}} - \mathbf{M}_1^{\mathbf{y}_{\overline{\mathbf{d}}}}\left(\widehat{\mathbf{M}}_1^{\mathbf{y}_{\overline{\mathbf{d}}}}\right)^{-1}\widehat{\mathbf{M}}_1^{\mathbf{y_d}}$, which is spanned by $\mathbf{q}_1 = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$. From (4.16) we then obtain $\overline{\mathbf{q}}_1 = (0,1,0)$. This leads us to band-limited susceptibilities

$$Q(x) = -\frac{1}{\sqrt{2}}Y_{1,-1}(x) - \frac{1}{\sqrt{2}}Y_{1,1}(x), \quad x \in \mathbb{S}, \tag{4.21}$$

$$\overline{Q}(x) = Y_{1,0}(x), \quad x \in \mathbb{S}. \tag{4.22}$$

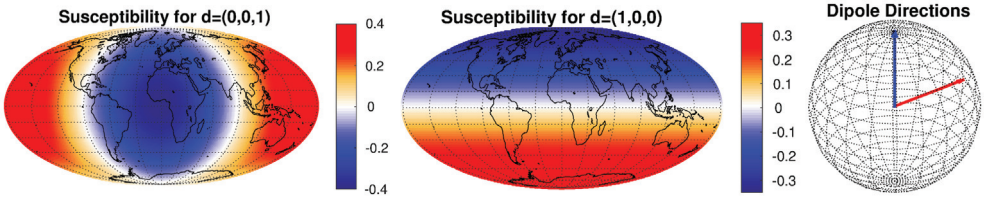

Figure 2. Illustration of the two band-limited dipole induced magnetizations with band-limit $N = 1$ described Remark 4.5 that are equivalent from outside. *Left*: susceptibility Q, *Center*: susceptibility $\overline{Q}$, *Right*: dipole directions $\mathbf{d}$ (blue) and $\overline{\mathbf{d}}$ (red).

They are illustrated in Figure 2. We see that, by the procedure described in the previous paragraphs, it is easy to construct band-limited $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$, with $\mathbf{d} \neq \pm\overline{\mathbf{d}}$, that are equivalent from outside. However, to check if, for a given $(Q, \mathbf{d})$, there exists another band-limited $\overline{Q}$ and $\overline{\mathbf{d}} \neq \pm\mathbf{d}$ such that $(Q, \mathbf{d})$ and $(\overline{Q}, \overline{\mathbf{d}})$ are equivalent from outside is somewhat more tedious. But essentially it boils down to finding zeros of polynomials.

To conclude this section, we summarize the previous considerations in the upcoming theorem. We actually formulate a slightly more general version that allows to decide if, for a given band-limited $\overline{\mathbf{m}} \in L^2(\mathbb{S}, \mathbb{R}^3)$ (not necessarily of dipole induced form (1.1)), there exists a dipole direction $\mathbf{d} \in \mathbb{S}$ and a susceptibility $Q \in L^2(\mathbb{S})$ such that $(Q, \mathbf{d})$ and $\overline{\mathbf{m}}$ are equivalent from outside. This is essentially a band-limited counterpart to Theorem 3.3.

Theorem 4.6: *Let $\overline{\mathbf{m}} \in L^2(\mathbb{S}, \mathbb{R}^3)$ be band-limited with band-limit $N + 1$. Then, there exists a band-limited susceptibility $Q \in L^2(\mathbb{S})$ and a dipole direction $\mathbf{d} \in \mathbb{S}$ such that $(Q, \mathbf{d})$ is equivalent to $\overline{\mathbf{m}}$ from outside if and only if there exists a vector $\mathbf{y} \in \mathbb{C}^3$ that is a zero of the vector-valued polynomial*

$$\mathbf{T}_{\overline{\mathbf{m}}} = \mathbf{R}^2_{N+1,\overline{\mathbf{m}}} + \mathbf{R}^2_{N,\overline{\mathbf{m}}} + \sum_{p=2}^{N-1} \mathbf{S}^2_{p,\overline{\mathbf{m}}},$$

and that can be written in the form $\mathbf{y} = \mathbf{y}_{\mathbf{d}} = (Y_{1,-1}(\mathbf{d}), Y_{1,0}(\mathbf{d}), Y_{1,1}(\mathbf{d}))$ for $\mathbf{d} \in \mathbb{S}$. The square in $\mathbf{R}^2_{p,\overline{\mathbf{m}}}$ and $\mathbf{S}^2_{p,\overline{\mathbf{m}}}$ is to be understood as acting componentwise on the vectors. The polynomials $\mathbf{R}_{p,\overline{\mathbf{m}}}$ and $\mathbf{S}_{p,\overline{\mathbf{m}}}$ are defined by

$$\mathbf{R}_{p,\overline{\mathbf{m}}}(\mathbf{y}) = \left(\Pi^p_{q=-p+2} c_q(p, \mathbf{y})\right) \left(\overline{\mathbf{m}}_p - \mathbf{M}^{\mathbf{y}}_{p-1} \left(\widehat{\mathbf{M}}^{\mathbf{y}}_{p-1}\right)^{-1} \widehat{\overline{\mathbf{m}}}_p\right),$$

$$\mathbf{S}_{p,\overline{\mathbf{m}}}(\mathbf{y}) = \left(\Pi^N_{n=p} \Pi^n_{q=-n+2} c_q(n, \mathbf{y})\right) \left(\overline{\mathbf{m}}_p - \mathbf{N}^{\mathbf{y}}_{p+1} \mathbf{q}_{p+1} - \mathbf{M}^{\mathbf{y}}_{p-1} \left(\widehat{\mathbf{M}}^{\mathbf{y}}_{p-1}\right)^{-1} \left(\widehat{\overline{\mathbf{m}}}_p - \widehat{\mathbf{N}}^{\mathbf{y}}_{p+1} \mathbf{q}_{p+1}\right)\right),$$

with $\overline{\mathbf{m}}_p = \left((\tilde{\overline{m}}^{(2)})^\wedge(p, -p), \ldots, (\tilde{\overline{m}}^{(2)})^\wedge(p, p)\right)$ and $\mathbf{q}_p = (Q^\wedge(p, -p), \ldots, Q^\wedge(p, p))$. The matrices $\mathbf{M}^{\mathbf{y}}_{p-1}$ and $\mathbf{N}^{\mathbf{y}}_{p+1}$ are given as in (4.9) and (4.10). $\widehat{\mathbf{M}}^{\mathbf{y}}_{p-1}$ and $\widehat{\mathbf{N}}^{\mathbf{y}}_{p+1}$ denote the matrices $\mathbf{M}^{\mathbf{y}}_{p-1}$ and $\mathbf{N}^{\mathbf{y}}_{p+1}$, respectively, with its first two rows (for $\mathbf{d} \in \mathbb{S} \setminus \{(0, 0, \pm 1)\}$) or its first and last row (for $\mathbf{d} = (0, 0, \pm 1)$) deleted. Analogously, $\widehat{\overline{\mathbf{m}}}_p$ represents the vector $\overline{\mathbf{m}}_p$ with its first two entries or its first and last entry deleted. The vectors $\mathbf{q}_p$, containing the Fourier coefficients of Q, can be computed iteratively by

$$\mathbf{q}_{p-1} = \left(\widehat{\mathbf{M}}^{\mathbf{y}}_{p-1}\right)^{-1} \widehat{\overline{\mathbf{m}}}_p, \quad p \in \{N, N+1\},$$

$$\mathbf{q}_{p-1} = \left(\widehat{\mathbf{M}}^{\mathbf{y}}_{p-1}\right)^{-1} \left(\widehat{\overline{\mathbf{m}}}_p - \widehat{\mathbf{N}}^{\mathbf{y}}_{p+1} \mathbf{q}_{p+1}\right), \quad p = N-1, \ldots, 2.$$

***Proof*:** The condition (4.14) for two dipole induced magnetizations can be rewritten in the following way

$$\mathbf{M}^{\mathbf{y}}_{p-1} \mathbf{q}_{p-1} + \mathbf{N}^{\mathbf{y}}_{p+1} \mathbf{q}_{p+1} - \overline{\mathbf{m}}_p = 0, \quad p \geq 1, \tag{4.23}$$

to fit the set-up of the theorem. The desired results then follow in the exact same manner as described in the previous paragraphs. The polynomial $\mathbf{T}_{\overline{\mathbf{m}}}$ has only been introduced to obtain a single non-negative

polynomial of which the zeros have to be found, rather than finding zeros separately for all $\mathbf{R}_{p,\overline{\mathbf{m}}}$ and $\mathbf{S}_{p,\overline{\mathbf{m}}}$. □

Remark 4.7: Just as mentioned in Remark 3.4, for a given magnetic potential V, one first has to find a general magnetization $\overline{\mathbf{m}}$ such that $V = V[\overline{\mathbf{m}}]$ on $\mathbb{S}_R$. Afterwards one can use Theorem 4.6 to check whether V can also be expressed in the form $V = V[Q, \mathbf{d}]$. For the construction of the polynomial $\mathbf{T}_{\overline{\mathbf{m}}}$ in Theorem 4.6, only the contribution $\tilde{\overline{\mathbf{m}}}^{(2)}$ of $\overline{\mathbf{m}}$, which is determined uniquely by V, is required.

5. Numerical Examples

We now provide some numerical examples based on the considerations for spatial localization constraints in Section 3.

5.1. The Algorithm

Remark 3.4 motivates the following three-step procedure to check whether a susceptibility Q and dipole direction $\mathbf{d} \in \mathbb{S}$ exist such that $V[Q, \mathbf{d}] = V$ for a given potential V on $\mathbb{S}_R$ and to actually compute such Q, $\mathbf{d}$. In fact, the focus is on finding a suitable dipole direction $\mathbf{d}$ (this is the quantity of interest, e.g. in some paleomagnetic problems; and once the dipole direction is known, the susceptibility could be obtained by solving the linear inverse problem $V[Q, \mathbf{d}] = V$ for a given $\mathbf{d}$).

Procedure 5.1: Let a magnetic potential V be given on a sphere $\mathbb{S}_R$ of radius $R > 1$, and let $\Gamma \subset \mathbb{S}$ be a subregion with $\overline{\Gamma} \neq \mathbb{S}$. Then proceed as follows:

(1) Find a magnetization $\mathbf{m}^* \in H_1(\mathbb{S}, \mathbb{R}^3)$ with $\operatorname{supp}(\mathbf{m}^*) \subset \Gamma$ that satisfies

$$V[\mathbf{m}^*](x) = V(x), \quad x \in \mathbb{S}_R.$$

By $V[\mathbf{m}]$ we denote the magnetic potential generated by $\mathbf{m}$ via (1.2).

(2) Compute g^*, $\mathbf{h}^*$, and m_1^* from the $\mathbf{m}^*$ obtained in (1). Find a $\mathbf{d}^* \in \mathbb{S}$ that satisfies

$$T_{g^*,\mathbf{h}^*,m_1^*}(\mathbf{d}^*) = \min_{\mathbf{d}\in\mathbb{S}} T_{g^*,\mathbf{h}^*,m_1^*}(\mathbf{d}).$$

(3) Find a susceptibility $Q^* \in H_1(\mathbb{S})$ with $\operatorname{supp}(Q^*) \subset \Gamma$ such that

$$V[Q^*, \mathbf{d}^*] = V(x), \quad x \in \mathbb{S}_R.$$

By $V[Q, \mathbf{d}]$ we denote the magnetic potential $V[\mathbf{m}]$ generated by a magnetization $\mathbf{m}$ of the form (1.1).

If the data misfit $\|V[Q^*, \mathbf{d}^*] - V\|_{L^2(\mathbb{S}_R)}$ in step (3) is 'too large', go back to (2), find a new $\mathbf{d}^{**} \neq \mathbf{d}^* \in \mathbb{S}$ and repeat step (3) with this $\mathbf{d}^{**}$. If no other $\mathbf{d}^{**}$ exists, this is an indicator that the given magnetic potential V cannot be produced by a dipole induced magnetization. If $\|V[Q^*, \mathbf{d}^*] - V\|_{L^2(\mathbb{S}_R)}$ in step (3) is 'sufficiently small', then Q^* and $\mathbf{d}^*$ represent a susceptibility and a dipole direction with the desired properties.

Remark 5.2: There are a few aspects of Procedure 5.1 that require further clarification and discussion:

(a) *If only interested in the dipole direction* $\mathbf{d}$: the first two steps in Procedure 5.1 are the important ones, with step (2) supplying the candidates $\mathbf{d}^*$ for $\mathbf{d}$. The corresponding susceptibility is directly given by $Q^*(x) = \frac{x\cdot\mathbf{m}^*(x)}{2x\cdot\mathbf{d}^*}$. However, Theorem 3.3 requires $Q^* \in H_1(\mathbb{S})$ in order to guarantee that $V[Q^*, \mathbf{d}^*] = V$. Due to measurement and reconstruction errors in V and $\mathbf{m}^*$ the latter is unlikely to be achieved in reality. Therefore, step (3) can be seen as a validation of the result of the first two steps. Inverting V again in step (3), now with a given $\mathbf{d}^*$, supplies an approximation Q^* of Q that lies in $H_1(\mathbb{S})$. If the data misfit $\|V[Q^*, \mathbf{d}^*] - V\|_{L^2(\mathbb{S}_R)}$ is 'small enough', this indicates that $\mathbf{d}^*$ is an admissible dipole direction.

(b) *Why the minimization in step (2)?*: Theorem 3.3 actually requires to find a zero $\mathbf{d}^*$ of $T_{g^*,\mathbf{h}^*,m_1^*}$. However, such a zero might not exist either because there does not exist a dipole induced

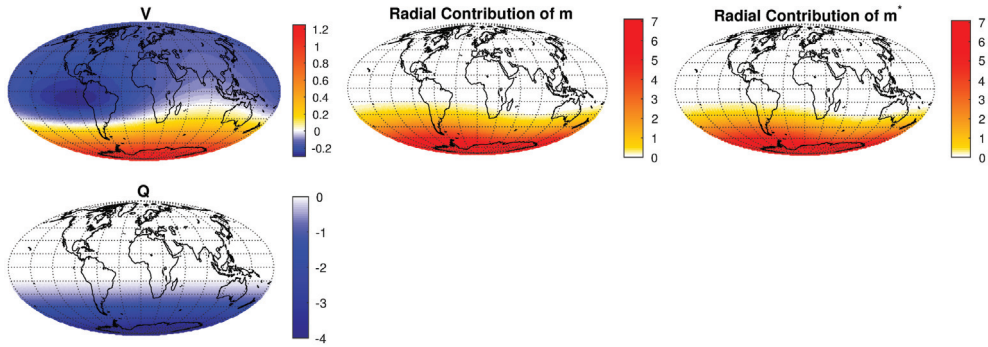

Figure 3. Illustration of step (1) for situation (a): noise-free input data V (*top left*), underlying susceptibility Q (*bottom left*), radial component m_1 of the true magnetization $\mathbf{m}$ (*center*, and radial component m_1^* of the reconstructed magnetization $\mathbf{m}^*$ (*right*).

magnetization that produces V in the first place or because noise in the measurements or reconstruction errors may have lead to a deteriorated version of $T_{g^*,\mathbf{h}^*,m_1^*}$. In order to exclude false conclusions due to the latter mentioned error sources, we minimize $T_{g^*,\mathbf{h}^*,m_1^*}$ instead of trying to find its zeros (since $T_{g^*,\mathbf{h}^*,m_1^*}$ is always non-negative by construction, this procedure is equivalent). The minimization of $T_{g^*,\mathbf{h}^*,m_1^*}$ is actually stable with respect to errors in the coefficients of the polynomial. The instability of the problem at hand is due to the ill-posedness of the inverse problem that has to be solved in step (1).

(c) *What is meant by 'too large' and 'too small'?*: As always in inverse problems, this is a delicate question. One possible criterion could be the quantity

$$\delta_{\mathbf{d}^*} = \frac{\|V[Q^*, \mathbf{d}^*] - V\|_{L^2(\mathbb{S}_R)}}{\|V\|_{L^2(\mathbb{S}_R)}},$$

motivated by Morozov's discrepancy principle (see, e.g. [23]). If $\delta_{\mathbf{d}^*}$ is about the same order as the data noise level ε, then we accept the candidate $\mathbf{d}^*$. However, a detailed analysis of this criterion is outside the scope of this article.

(d) Last but not least, it should be noted that the inverse problems in step (1) and (3) of Procedure 5.1 are linear (opposed to computing approximations Q^* and $\mathbf{d}^*$ directly from a single inversion of V). Additionally, Procedure 5.1 supplies more information on possible candidates for dipole directions than the direct inversion.

5.2. The examples

We illustrate Procedure 5.1 for three different situations. All situations have in common that the potential V is given on $\mathbb{S}_R$, with $R = 1.06$ (which simulates the situation of a satellite flying at an altitude of around 380km above the Earth's surface). Furthermore, V is assumed to be given only in discrete points on an equiangular grid of 40, 401 points. The magnetization $\mathbf{m}$ on $\mathbb{S}$ that generates V is varied among the three situations, but it is always supported in the lower hemisphere, i.e. $\mathrm{supp}(\mathbf{m}) \subset \Gamma = \{x \in \mathbb{S} : x \cdot \mathbf{v} \leq 0\}$ for $\mathbf{v} = (0, 0, 1)^T$ being fixed:

(a) $\mathbf{m}$ is a dipole induced magnetization that is uniquely determined. In particular, $\mathbf{m}$ is of the form (1.1) with dipole direction $\mathbf{d} = (0, 0.436, 0.9)^T$ and susceptibility

$$Q(x) = 4(x \cdot \mathbf{v})^3 \chi_{[-1,0]}(x \cdot \mathbf{v}), \quad x \in \mathbb{S},$$

where $\chi_{[-1,0]}$ denotes the characteristic function on the interval $[-1, 0]$.

(a') Same as in (a) but only a noisy version V^ε of V is given. In this example, we choose the noise level $\varepsilon = \|V^\varepsilon - V\|_{L^2(\mathbb{S}_R)} / \|V\|_{L^2(\mathbb{S}_R)} = 10^{-2}$.

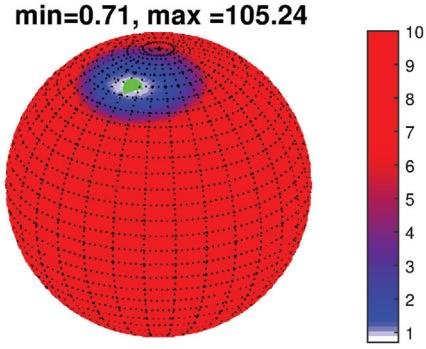

Figure 4. Illustration of step (2) for situation (a): the figure shows the evaluation of $T_{g^*,\mathbf{h}^*,m_1^*}$ on the unit sphere, the *green dot* indicates the location of the true dipole direction $\mathbf{d}$. The color bar has been modified to emphasize the minimum, the actual minimum and maximum is indicated in the title.

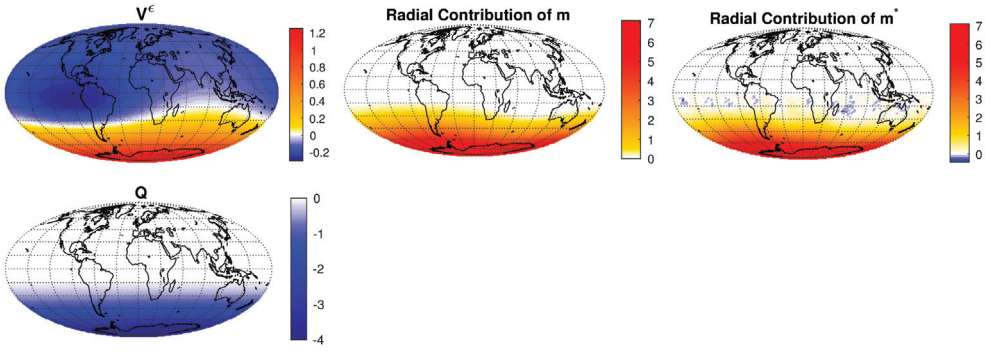

Figure 5. Illustration of step (1) for situation (a′): noisy input data V (*top left*), underlying susceptibility Q (*bottom left*), radial component m_1 of the true magnetization $\mathbf{m}$ (*center*), and radial component m_1^* of the reconstructed magnetization $\mathbf{m}^*$ (*right*).

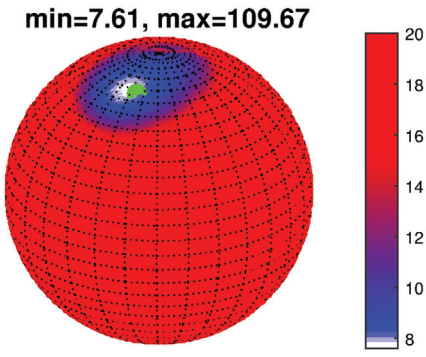

Figure 6. Illustration of step (2) for situation (a′): the figure shows the evaluation of $T_{g^*,\mathbf{h}^*,m_1^*}$ on the unit sphere, the *green dot* indicates the location of the true dipole direction $\mathbf{d}$. The color bar has been modified to emphasize the minimum, the actual minimum and maximum is indicated in the title.

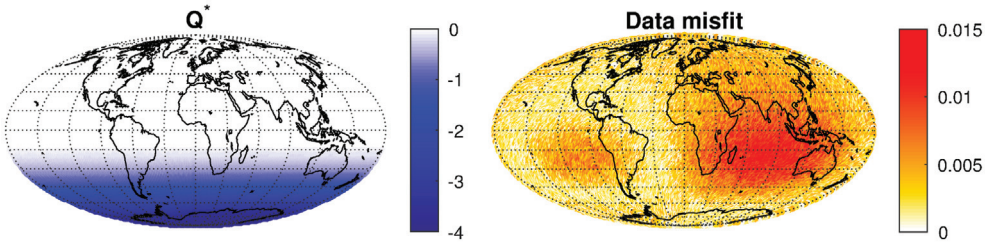

Figure 7. Illustration of step (3) for situation (a′): reconstructed susceptibility Q^* for $\mathbf{d}^* = (0.027, 0.433, 0.901)^T$ (*left*), data misfit $|V^\varepsilon - V[Q^*, \mathbf{d}^*]|$ (*right*).

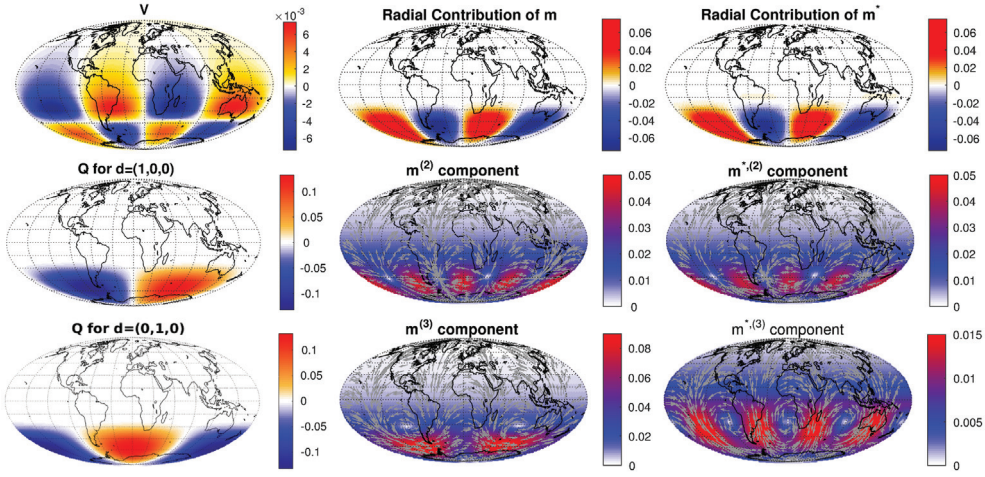

Figure 8. Illustration of step (1) for situation (b): input data V (*top left*), underlying equivalent susceptibilities Q (*center and bottom left*), radial component m_1 and the contributions $\mathbf{m}^{(2)}$ and $\mathbf{m}^{(3)}$ of the true magnetization $\mathbf{m}$ (*center*), and radial component m_1^* and the contributions $\mathbf{m}^{*,(2)}$ and $\mathbf{m}^{*,(3)}$ of the reconstructed magnetization $\mathbf{m}^*$ (*right*). In the plots of the second and third row, colors indicate the absolute values $|\mathbf{m}^{(i)}|$ and $|\mathbf{m}^{*,(i)}|$, $i = 1, 2$, and arrows the orientation.

(b) $\mathbf{m}$ is a dipole induced magnetization that is non-unique and of a form as described in Remark 3.2. In particular, we choose the dipole direction $\mathbf{d} = (1, 0, 0)^T$ and the susceptibility

$$Q(x) = P\left(x \cdot \frac{\mathbf{d} \times \bar{\mathbf{d}}}{|\mathbf{d} \times \bar{\mathbf{d}}|}\right) x \cdot \bar{\mathbf{d}}, \quad x \in \mathbb{S}, \qquad P(t) = e^{-\frac{1}{t^2}} \chi_{[-1,0]}(t), \quad t \in [-1, 1],$$

where $\bar{\mathbf{d}} = (0, 1, 0)^T$ is a fixed auxiliary vector. (According to Remark 3.2, choosing $\bar{Q}(x) = P\big(x \cdot (\mathbf{d} \times \bar{\mathbf{d}})/|\mathbf{d} \times \bar{\mathbf{d}}|\big) x \cdot \mathbf{d}$ yields a further dipole induced magnetization that is equivalent to $\mathbf{m}$ from above. In other words, $(\bar{Q}, \bar{\mathbf{d}})$ is equivalent from above to $(Q, \mathbf{d})$.)

(c) $\mathbf{m}$ is not a dipole induced magnetization. In particular, we choose

$$\mathbf{m}(x) = Q(x)\mathbf{v}, \quad x \in \mathbb{S},$$

with Q as in (a).

For each of the situations above we apply the first two steps of Procedure 5.1 (the third step is only indicated for situation (a')). In step (1), we construct $\mathbf{m}^*$ to be the minimizer of the functional

$$\mathcal{F}[\mathbf{m}] = \|V[\mathbf{m}] - V\|^2_{L^2(\mathbb{S}_R)} + \alpha \|\mathbf{m}\|^2_{H_1(\mathbb{S},\mathbb{R}^3)} + \beta \|\mathbf{m}\|^2_{L^2(\mathbb{S}\setminus\Gamma,\mathbb{R}^3)}, \tag{5.1}$$

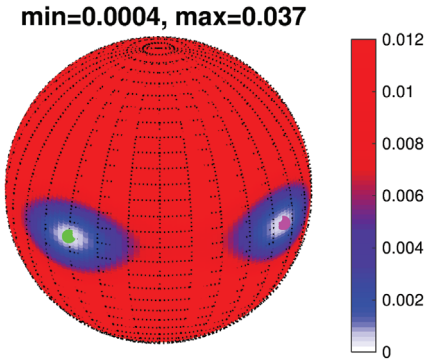

Figure 9. Illustration of step (2) for situation (b): the figure shows the evaluation of $T_{g^*,\mathbf{h}^*,m_1^*}$ on the unit sphere, the *green and purple dots* indicate the locations of the possible (true) dipole directions $\mathbf{d}$ and $\bar{\mathbf{d}}$, respectively. The color bar has been modified to emphasize the minimum, the actual minimum and maximum is indicated in the title.

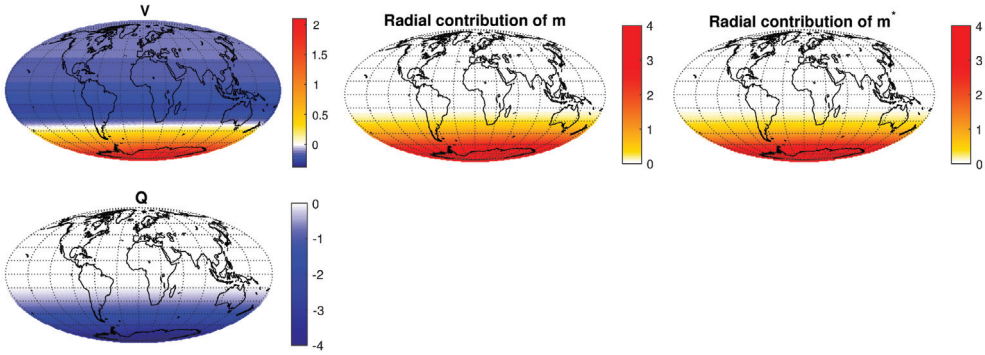

Figure 10. Illustration of step (1) for situation (c): input data V (*top left*), underlying susceptibility (*bottom left*), radial component m_1 of the true magnetization $\mathbf{m}$ (*center*), and radial component m_1^* of the reconstructed magnetization $\mathbf{m}^*$ (*right*).

where $\|\mathbf{m}\|_{H_1(\mathbb{S},\mathbb{R}^3)}$ denotes the Sobolev norm of $\mathbf{m}$ (see, e.g. [16] for more details). The first term in (5.1) simply represents a data misfit that measures the deviation of $V[\mathbf{m}]$ from the known magnetic potential V, while the second term is a Tikhonov-type regularization to reduce noise amplification resulting from the ill-posedness of the downward continuation of the potential field data V to the surface $\mathbb{S}$ (this is well-studied and can be found, e.g. in [24,25] and references therein). The third term in (5.1) eventually penalizes magnetizations $\mathbf{m}$ that have contributions outside Γ, i.e. magnetizations that do not satisfy $\mathrm{supp}(\mathbf{m}) \subset \Gamma$. For the discretization of $\mathcal{F}[\mathbf{m}]$, we expand $\mathbf{m}$ in terms of (vectorial) Abel-Poisson kernels:

$$\mathbf{m}(x) = \sum_{i=1}^{3} \sum_{n=1}^{N} \gamma_{i,n}\, o_x^{(i)} K(x \cdot x_n), \tag{5.2}$$

$$K(x \cdot x_n) = \frac{1 - h^2}{(1 + h^2 - 2h(x \cdot x_n))^{\frac{3}{2}}}, \tag{5.3}$$

where $h \in (0,1)$ is a fixed parameter (influencing the localization of K; we use $h = 0.9$) and $\{x_n\}_{n=1,\dots,N} \subset \mathbb{S}$ is a set of uniformly distributed points indicating the centers of the kernel K (in our case, we choose $N = 10235$ different centers). Some general properties of the Abel-Poisson kernel K can be found, e.g. in [14]. With this discretization, the minimization of $\mathcal{F}[\mathbf{m}]$ reduces to solving a set of linear equations with

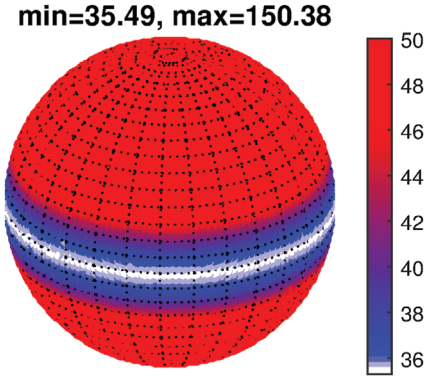

Figure 11. Illustration of step (2) for situation (c): the figure shows the evaluation of $T_{g^*,\mathbf{h}^*,m_1^*}$ on the unit sphere. The color bar has been modified to emphasize the minimum, the actual minimum and maximum is indicated in the title.

respect to the coefficients $\gamma_{i,n}$. In step (2), we compute $T_{g^*,\mathbf{h}^*,m_1^*}$ from the $\mathbf{m}^*$ obtained in step (1). For the purpose of illustration, we simply plot $T_{g^*,\mathbf{h}^*,m_1^*}$ over the sphere to indicate where the minima $\mathbf{d}^* \in \mathbb{S}$ are located. Eventually, given $\mathbf{d}^* \in \mathbb{S}$, in step (3) we minimize a functional $\mathcal{G}$ similar to (5.1) in order to obtain Q^*. More precisely, we minimize

$$\mathcal{G}[Q] = \left\| V[Q, \mathbf{d}^*] - V \right\|^2_{L^2(\mathbb{S}_R)} + \alpha \|\mathbf{m}[Q]\|^2_{H_1(\mathbb{S},\mathbb{R}^3)} + \beta \|\mathbf{m}[Q]\|^2_{L^2(\mathbb{S}\setminus\Gamma,\mathbb{R}^3)}, \tag{5.4}$$

where $\mathbf{m}[Q]$ denotes the induced magnetization $\mathbf{m}[Q](x) = Q(x)(3(x \cdot \mathbf{d}^*)x - \mathbf{d}^*)$, $x \in \mathbb{S}$. For the numerical evaluation, we proceed similarly as for (5.1) by expanding Q in terms of (scalar) Abel Poisson kernels and solving a corresponding system of linear equations (details for a similar problem can be found in [2]). Any numerical integrations necessary during the procedure are performed via the methods of [26] (when the integration region comprises the entire sphere $\mathbb{S}$ or $\mathbb{S}_R$, respectively) and [27] (when the integration is only performed over a spherical cap $\mathbb{S} \setminus \Gamma$).

The magnetizations involved in examples (a)–(c) are all localized in the Southern hemisphere. Choosing magnetizations with a larger support typically makes the numerical solution of the problem more delicate. However, as indicated in Remark 5.2(b), this instability affects mainly the inverse problems in steps (1) and (3) of Procedure 5.1. Numerical examples for these problems with susceptibilities covering larger portions of the sphere can be found, e.g. in [2]. In the paper at hand we want to focus on the reconstruction of dipole directions and stick to a single set-up for the localization of the magnetizations.

The results of step (1) and (2) for situation (a) are indicated in Figures 3 and 4, respectively. The reconstruction $\mathbf{m}^*$ nicely fits the true $\mathbf{m}$. For brevity, we illustrated only the radial components. Figure 4 shows that the minimum of $T_{g^*,\mathbf{h}^*,m_1^*}$ coincides with the true dipole direction $\mathbf{d}$ (which is indicated by a green dot). The corresponding results for the noisy situation (a') are indicated in Figures 5 and 6. We see that the reconstructed radial contribution of $\mathbf{m}^*$ shows some minor artifacts but the dipole direction $\mathbf{d}$ still coincides quite well with the minimum of $T_{g^*,\mathbf{h}^*,m_1^*}$. However, we see that the actual minimum value is rather large in the noisy set-up. Therefore, to make sure that we found an admissible candidate $\mathbf{d}^*$ for the dipole direction, we proceed to step (3) with the approximation $\mathbf{d}^* = (0.027, 0.433, 0.901)^T$ of the minimum of $T_{g^*,\mathbf{h}^*,m_1^*}$. The reconstructed susceptibility Q^* is indicated in Figure 7 and matches the true susceptibility Q. The data misfit $|V^\varepsilon - V[Q^*, \mathbf{d}^*]|$ allows to compute the quantity $\delta_{\mathbf{d}^*}$ from Remark 5.2(c) and does not require the knowledge of Q. In this case, we have $\delta_{\mathbf{d}^*} = 0.011$, leading us to accept $\mathbf{d}^*$ as an approximation of the true dipole direction.

Steps (1) and (2) for situation (b), where no uniqueness of Q and $\mathbf{d}$ is given, are shown in Figures 8 and 9, respectively. In Figure 8, we indicated all three contributions (i.e. the radial contribution m_1 and the surface curl- and surface divergence-free contributions $\mathbf{m}^{(2)}$ and $\mathbf{m}^{(3)}$, respectively) of $\mathbf{m}$ and $\mathbf{m}^*$. It is seen that the radial contribution and the surface curl-free contribution of the true and the reconstructed magnetization coincide, as is expected from Theorem 2.4. However, the surface divergence-

free contribution is not uniquely determined and therefore may differ, as is the case here. But the latter has no impact on our further procedure. Figure 9 shows that the two possible (true) dipole directions $\mathbf{d} = (1,0,0)^T$ and $\overline{\mathbf{d}} = (0,1,0)^T$ are located at the minima of $T_{g^*,\mathbf{h}^*,m_1^*}$ ($\mathbf{d}$ and $\overline{\mathbf{d}}$ are indicated by green and purple dots). Which direction is the correct one cannot be decided without further a priori geophysical information due to the intrinsic non-uniqueness.

For situation (c), the magnetization $\mathbf{m}$ has been reconstructed very well as can be exemplarily seen for the radial component in Figure 10. Figure 11 shows that the minima of $T_{g^*,\mathbf{h}^*,m_1^*}$ are located on the equator, i.e. any possible candidate for a dipole direction $\mathbf{d}^*$ must lie in the equatorial plane. However, although no noise was included in the data, the acquired minimum value of $T_{g^*,\mathbf{h}^*,m_1^*}$ deviates so significantly from zero that this leads us to conclude that the potential V cannot be generated by a dipole induced magnetization.

6. Conclusion

The fact that generally only the $\tilde{\mathbf{m}}^{(2)}$-contribution of a spherical magnetization $\mathbf{m}$ can be uniquely reconstructed from satellite magnetic field measurements leads to uniqueness issues, e.g. in determining possible dipole directions (assuming that the underlying magnetization is of induced type). The additional assumption that $\mathbf{m}$ is localized in some subregion of a spherical planetary surface allows to uniquely determine the $\tilde{\mathbf{m}}^{(1)}$- and $\tilde{\mathbf{m}}^{(2)}$-contributions of $\mathbf{m}$ (although $\tilde{\mathbf{m}}^{(3)}$ is still unknown), which implies that the radial contribution $\mathbf{m}^{(1)}$ and the tangential surface curl-free contribution $\mathbf{m}^{(2)}$ are determined uniquely. Here, we have shown that for the latter situation there exists a procedure for the determination of candidates for the dipole direction $\mathbf{d}$ and for the decision if a measured magnetic field can be produced by a dipole induced magnetization in the first place (a similar procedure has been derived for band-limited magnetizations, but in our examples in Section 5 we focused on the spatial localization constraint as we believe it to be more feasible for actual applications). The numerical treatment of the involved extremal problems allows various approaches and should be investigated in more detail for future applications. Also the cooperation with geophysicists will be required in order to embed the proposed method into a scenario applicable for real geophysical problems (such as paleopole estimations). The focus of the paper in its current form is on the presentation and illustration of the conceptual set-up for the improved reconstruction of dipole directions and the investigation of uniqueness issues.

Acknowledgements

The author thanks Foteini Vervelidou, GFZ Potsdam, for pointing out the problem of studying the reconstruction of dipole directions. Furthermore, the author would like to thank the anonymous reviewers for their helpful remarks and suggestions for the improvement of the paper.

Disclosure statement

No potential conflict of interest was reported by the author.

Funding

The work was partly supported by DFG [grant GE 2781/1-1].

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